Lecture 8

Global Cauchy theorem

MATH 503, FALL 2025

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Chains and cycles

- Suppose $\gamma_1, \ldots, \gamma_n$ are paths in $\mathbb{C}$, and set $K = \gamma_1^* \cup \cdots \cup \gamma_n^*$. Let $C(K)$ be the vector space of continuous functions on K. Each γ_i induces a linear functional $\tilde{\gamma}_i$ on $C(K)$, by the formula

$$\tilde{\gamma}_i(f) = \int_{\gamma_i} f(z)dz.$$

- Define $\tilde{\Gamma} = \tilde{\gamma}_1 + \cdots + \tilde{\gamma}_n$ or more explicitly,

$$\tilde{\Gamma}(f) = \tilde{\gamma}_1(f) + \cdots + \tilde{\gamma}_n(f), \quad \text{for} \quad f \in C(K). \tag{*}$$

- The relation (*) suggests that we introduce a "formal sum"

$$\Gamma = \gamma_1 \dot{+} \cdots \dot{+} \gamma_n,$$

and define

$$\int_\Gamma f(z)dz = \tilde{\Gamma}(f) = \sum_{i=1}^{n} \int_{\gamma_i} f(z)dz, \quad \text{for} \quad f \in C(K).$$

Chains and cycles

Let $\gamma_1, \ldots, \gamma_n$ be paths such that $\gamma_i^* \subset \Omega$ for $1 \leq i \leq n$.

- Let Γ be a formal sum as before given by

$$\Gamma = \gamma_1 \dot{+} \cdots \dot{+} \gamma_n. \tag{**}$$

  Then we say that Γ is a **chain** in Ω.

- If there exists a representation of a chain Γ such that each γ_i is a closed path in Ω, then we say that Γ is a **cycle** in Ω.
- By a combinatorial argument, it can be shown that a chain Γ is a cycle if and only if in any representation of Γ, the initial and end points of γ_i are identical in pairs.
- If (**) holds, then we define $\Gamma^* = \gamma_1^* \cup \cdots \cup \gamma_n^*$.
- The relation (**) means that we are adding paths in the context of adding linear functionals (*); otherwise, it would not be meaningful.

Chains and cycles

- If Γ is a cycle and $\alpha \notin \Gamma^*$, we define the index of α with respect to Γ by
$$\operatorname{Ind}_\Gamma(\alpha) = \frac{1}{2\pi i} \int_\Gamma \frac{dz}{z-\alpha}.$$
- Obviously, (**) implies
$$\operatorname{Ind}_\Gamma(\alpha) = \sum_{i=1}^{n} \operatorname{Ind}_{\gamma_i}(\alpha).$$
- If each γ_i in (**) is replaced by its opposite path $-\gamma_i$, the resulting chain will be denoted by $-\Gamma$. Then
$$\int_{-\Gamma} f(z)dz = -\int_\Gamma f(z)dz \quad \text{for} \quad f \in C\left(\Gamma^*\right).$$
- In particular, $\operatorname{Ind}_{-\Gamma}(\alpha) = -\operatorname{Ind}_\Gamma(\alpha)$ if Γ is a cycle and $\alpha \notin \Gamma^*$.

Chains and cycles

- Chains can be added and subtracted in the obvious way, by adding or subtracting the corresponding functionals.
- The statement $\Gamma = k_1\Gamma_1 \dot{+} k_2\Gamma_2$ for any integers $k_1, k_2 \in \mathbb{Z}$ means that

$$\int_\Gamma f(z)dz = k_1 \int_{\Gamma_1} f(z)dz + k_2 \int_{\Gamma_2} f(z)dz \quad \text{for all} \quad f \in C\left(\Gamma_1^* \cup \Gamma_2^*\right).$$

- Finally, note that a chain may be represented as a sum of paths in many ways. To say that

$$\gamma_1 \dot{+} \cdots \dot{+} \gamma_n = \delta_1 \dot{+} \cdots \dot{+} \delta_k$$

means simply that

$$\sum_{i=1}^{n} \int_{\gamma_i} f(z)dz = \sum_{j=1}^{k} \int_{\delta_j} f(z)dz$$

for every f that is continuous on $\gamma_1^* \cup \cdots \cup \gamma_n^* \cup \delta_1^* \cup \cdots \cup \delta_k^*$.
- In particular, a cycle may very well be represented as a sum of paths that are not closed.

Global Cauchy's theorem

Theorem

Suppose $f \in H(\Omega)$, *where* Ω *is an arbitrary open set in the complex plane.*

- *If* Γ *is a cycle in* Ω *that satisfies* $\mathrm{Ind}_\Gamma(\alpha) = 0$ *for every* $\alpha \notin \Omega$, *then*

$$f(z) \cdot \mathrm{Ind}_\Gamma(z) = \frac{1}{2\pi i} \int_\Gamma \frac{f(w)}{w - z} dw \quad \text{for} \quad z \in \Omega \setminus \Gamma^*, \tag{A}$$

  and

$$\int_\Gamma f(z) dz = 0, \tag{B}$$

- *If* Γ_0 *and* Γ_1 *are cycles in* Ω *such that* $\mathrm{Ind}_{\Gamma_0}(\alpha) = \mathrm{Ind}_{\Gamma_1}(\alpha)$ *for every* $\alpha \notin \Omega$, *then*

$$\int_{\Gamma_0} f(z) dz = \int_{\Gamma_1} f(z) dz. \tag{C}$$

Global Cauchy's theorem

Proof: The function g defined in $\Omega \times \Omega$ by

$$g(z,w) = \begin{cases} \frac{f(w)-f(z)}{w-z} & \text{if } w \neq z, \\ f'(z) & \text{if } w = z, \end{cases}$$

is continuous in $\Omega \times \Omega$ (see the previous lecture).

- Hence we can define

$$h(z) = \frac{1}{2\pi i} \int_\Gamma g(z,w)dw \quad \text{for} \quad z \in \Omega.$$

- For $z \in \Omega \setminus \Gamma^*$, the Cauchy formula (A) is clearly equivalent to the assertion that

$$h(z) = 0.$$

- To prove that $h(z) = 0$, let us first prove that $h \in H(\Omega)$.

Global Cauchy's theorem

- We first show that h is continuous in Ω.
- Observe that g is uniformly continuous on every compact subset of $\Omega \times \Omega$. If $z \in \Omega$ and $(z_n)_{n\in\mathbb{N}} \subseteq \Omega$, and $\lim_{n\to\infty} z_n = z$, it follows that $\overline{\{z_n : n \in \mathbb{N}\}} \times \Gamma^*$ is a compact subset of $\Omega \times \Omega$, and consequently $\lim_{n\to\infty} g(z_n, w) = g(z, w)$ uniformly for $w \in \Gamma^*$.
- Hence $\lim_{n\to\infty} h(z_n) = h(z)$, proving that h is continuous in Ω.
- Let Δ be a closed triangle in Ω. Then by Fubini's theorem

$$\int_{\partial\Delta} h(z)dz = \frac{1}{2\pi i}\int_{\Gamma}\left(\int_{\partial\Delta} g(z,w)dz\right)dw.$$

- For each $w \in \Omega$ the function $z \mapsto g(z, w)$ is holomorphic in Ω, since he singularity at $z = w$ is removable.
- The inner integral over $\partial\Delta$ is therefore 0 for every $w \in \Gamma^*$. Thus Morera's theorem shows now that $h \in H(\Omega)$ as desired.

Global Cauchy's theorem

- Let $\Omega_1 = \{z \in \mathbb{C} : \mathrm{Ind}_\Gamma(z) = 0\}$. Then $\Omega^c \subseteq \Omega_1$, since $\mathrm{Ind}_\Gamma(\alpha) = 0$ for all $\alpha \in \Omega^c$. Moreover, Ω_1 is open since the index function $\mathrm{Ind}_\Gamma : \mathbb{C} \setminus \Gamma^* \to \mathbb{Z}$ is continuous. Also Ω_1 contains the unbounded component of the complement of Γ^*, since $\mathrm{Ind}_\Gamma(z)$ is 0 there.
- Define

$$h_1(z) = \frac{1}{2\pi i} \int_\Gamma \frac{f(w)}{w - z} dw \quad \text{for} \quad z \in \Omega_1.$$

- If $z \in \Omega \cap \Omega_1$, the definition of Ω_1 makes it clear that $h_1(z) = h(z)$.
- Hence there is a function $\varphi \in H(\Omega \cup \Omega_1)$ whose restriction to Ω is h and whose restriction to Ω_1 is h_1.
- Since $\Omega \cup \Omega_1 = \mathbb{C}$, thus φ is an entire function. Hence

$$\lim_{|z| \to \infty} \varphi(z) = \lim_{|z| \to \infty} h_1(z) = 0.$$

- Liouville's theorem implies now that $\varphi(z) = 0$ for every $z \in \mathbb{C}$. This proves that $h(z) = 0$, and hence (A).

Global Cauchy's theorem

- To deduce (B) from (A), pick $a \in \Omega \setminus \Gamma^*$ and define

$$F(z) = (z - a)f(z).$$

Since $F(a) = 0$, and using (A) we obtain

$$\frac{1}{2\pi i}\int_{\Gamma} f(z)dz = \frac{1}{2\pi i}\int_{\Gamma} \frac{F(z)}{z-a}dz = F(a)\cdot \mathrm{Ind}_{\Gamma}(a) = 0.$$

- To prove (C) let $\Gamma = \Gamma_1 - \Gamma_0$. By our assumptions $\mathrm{Ind}_{\Gamma}(\alpha) = 0$ for every $\alpha \in \Omega^c$, hence by (B) we obtain

$$\int_{\Gamma} f(z)dz = 0.$$

- This is equivalent to statement (C):

$$\int_{\Gamma_0} f(z)dz = \int_{\Gamma_1} f(z)dz.$$

since $\Gamma = \Gamma_1 - \Gamma_0$. This completes the proof. □

Global Cauchy's theorem

Corollary

Let $\Omega \subseteq \mathbb{C}$ be an open set and $f \in H(\Omega)$, and let Γ be a cycle in Ω. Then the following statements are equivalent:

(i)
$$f(z) \cdot \mathrm{Ind}_\Gamma(z) = \frac{1}{2\pi i} \int_\Gamma \frac{f(w)}{w-z} dw \quad \text{for} \quad z \in \Omega \setminus \Gamma^*.$$

(ii)
$$\int_\Gamma f(z) dz = 0.$$

(iii)
$$\mathrm{Ind}_\Gamma(\alpha) = 0 \quad \text{for every} \quad \alpha \in \mathbb{C} \setminus \Omega.$$

Global Cauchy's theorem

Proof (i) $\implies$ (ii): Pick $a \in \Omega \setminus \Gamma^*$ and define

$$F(z) = (z-a)f(z).$$

Since $F(a) = 0$, and using (i) we obtain

$$\frac{1}{2\pi i}\int_\Gamma f(z)dz = \frac{1}{2\pi i}\int_\Gamma \frac{F(z)}{z-a}dz = F(a)\cdot \mathrm{Ind}_\Gamma(a) = 0. \qquad \square$$

Proof (ii) $\implies$ (iii): For $a \in \mathbb{C} \setminus \Omega$ the function $f(z) = (z-a)^{-1}$ is holomorphic in Ω. By (ii), since $\Gamma \subseteq \Omega$, we obtain

$$\mathrm{Ind}_\Gamma(a) = \frac{1}{2\pi i}\int_\Gamma \frac{1}{z-a}dz = 0.$$

Proof (iii) $\implies$ (i): This implication follows from the global Cauchy theorem. This completes the proof of the corollary. $\square$

Global Cauchy's theorem

Remarks

(a) If γ is a closed path in a convex region Ω and if $\alpha \notin \Omega$, an application of the Cauchy theorem for convex sets to $f(z) = (z - \alpha)^{-1}$ shows that

$$\mathrm{Ind}_\gamma(\alpha) = 0.$$

- Therefore the hypothesis of the global Cauchy theorem is therefore satisfied by every cycle in Ω if Ω is convex.
- This shows that the global Cauchy theorem generalizes Cauchy theorem and the Cauchy integral theorem for convex regions.

Global Cauchy's theorem

Remarks

(b) In order to apply the global Cauchy theorem, it is desirable to have a reasonably efficient method of finding the index of a point with respect to a closed path. Here the concept of the winding number, which coincides with the index function for closed paths will help.

- Let $\gamma : [a, b] \to \mathbb{C}$ be a closed path and $z_0 \notin \gamma^*$. Suppose that θ_{z_0} is a continuous argument of $\gamma - z_0$. We recall that the **winding number** of z_0 with respect to γ, is defined by

$$W(\gamma, z_0) = \frac{\theta_{z_0}(b) - \theta_{z_0}(a)}{2\pi}.$$

- It was shown last time that

$$W(\gamma, z_0) = \frac{1}{2\pi i} \int_\gamma \frac{1}{w - z_0} dw = \mathrm{Ind}_\gamma(z_0).$$

Global Cauchy's theorem

Remarks

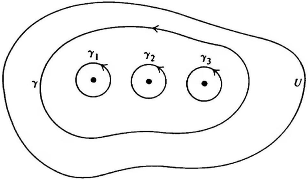
- The winding number $W(\gamma, z_0)$ computes precisely the number of times γ loops around z_0. Since $W(\gamma, z_0)$ is independent of the choice of continuous argument, we can analyze the change in argument of the quantity $w - z_0$ as w travels along γ.
- Each time γ loops around z_0 in an **anticlockwise direction**, then $\frac{1}{2\pi}\arg(w - z_0)$ **increases** by 1. Conversely, if γ loops around z_0 in a **clockwise direction**, then $\frac{1}{2\pi}\arg(w - z_0)$ **decreases** by 1.

(c) The last part of the global Cauchy theorem shows under what circumstances integration over one cycle can be replaced by integration over another, without changing the value of the integral.

Global Cauchy's theorem, examples

Example 1

- Let γ be a closed path, as below, in an open set $U \subseteq \mathbb{C}$, and let f be holomorphic on U.



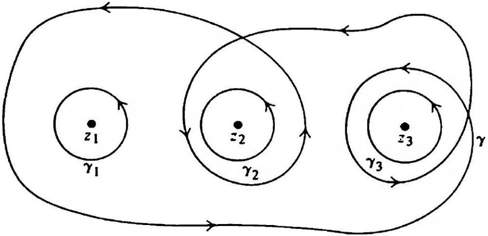
- In that figure, we see that γ winds around the three points z_1, z_2, z_3, and winds once, hence $\text{Ind}_\gamma(z_1) = \text{Ind}_\gamma(z_2) = \text{Ind}_\gamma(z_3) = 1$.
- Let $\gamma_1, \gamma_2, \gamma_3$ be small circles centered at z_1, z_2, z_3 respectively, and oriented anticlockwise. Then we see that

$$\int_\gamma f(z)dz = \int_{\gamma_1} f(z)dz + \int_{\gamma_2} f(z)dz + \int_{\gamma_3} f(z)dz.$$

Global Cauchy's theorem, examples

Example 2

- Let γ be the curve illustrated below, and let U be the plane from which three points z_1, z_2, z_3 have been deleted. Let $\gamma_1, \gamma_2, \gamma_3$ be small circles centered at z_1, z_2, z_3 respectively, oriented counterclockwise.



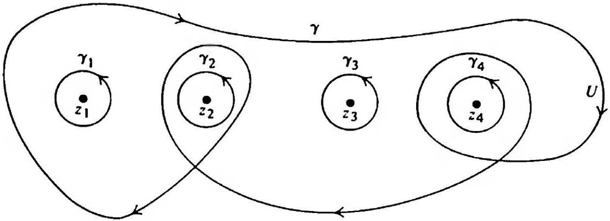
- Then $\text{Ind}_\gamma(z_1) = 1$ and $\text{Ind}_\gamma(z_2) = \text{Ind}_\gamma(z_3) = 2$ and consequently

$$\int_\gamma f(z)dz = \int_{\gamma_1} f(z)dz + 2\int_{\gamma_2} f(z)dz + 2\int_{\gamma_3} f(z)dz.$$

Global Cauchy's theorem, examples

Example 3

- Let γ be the curve illustrated below, and let U be the plane from which four points z_1, z_2, z_3, z_4 have been deleted. Let $\gamma_1, \gamma_2, \gamma_3, \gamma_4$ be small circles centered at z_1, z_2, z_3, z_4 respectively, oriented clockwise.



- Then $\mathrm{Ind}_\gamma(z_1) = \mathrm{Ind}_\gamma(z_3) = -1$ and $\mathrm{Ind}_\gamma(z_2) = \mathrm{Ind}_\gamma(z_4) = -2$ and consequently

$$\int_\gamma f(z)dz = -\int_{\gamma_1} f(z)dz - 2\int_{\gamma_2} f(z)dz - \int_{\gamma_3} f(z)dz - 2\int_{\gamma_4} f(z)dz.$$

Global Cauchy's theorem, example

In the proof of Laurent series representation we used the following form of the Cauchy integral formula:

Theorem

Let f be analytic on an open set Ω containing the annulus $\overline{A}(z_0, r_1, r_2)$ for $0 < r_1 < r_2 < \infty$, and let γ_1 and γ_2 denote the positively oriented inner and outer boundaries of the annulus. Then for $z \in A(z_0, r_1, r_2)$, we have

$$f(z) = \frac{1}{2\pi i}\int_{\gamma_2} \frac{f(w)}{w-z}dw - \frac{1}{2\pi i}\int_{\gamma_1} \frac{f(w)}{w-z}dw$$

Proof: The proof is a consequence of the global Cauchy theorem applied to the cycle $\Gamma = \gamma_2 - \gamma_1$, since

$$\mathrm{Ind}_\Gamma(\alpha) = 0 \quad \text{for} \quad \alpha \notin A(z_0, \rho_1, \rho_2),$$

where $0 < \rho_1 < r_1 < r_2 < \rho_2 < \infty$ satisfy $\overline{A}(z_0, \rho_1, \rho_2) \subset \Omega$. □

Homotopy

Definition

Suppose γ_0 and γ_1 are closed curves in a topological space X, both with parameter interval $I = [0, 1]$. We say that γ_0 and γ_1 are X-**homotopic** if there is a continuous mapping H of the unit square $I \times I$ into X such that

$$\begin{aligned} H(s,0) = \gamma_0(s), \quad H(s,1) = \gamma_1(s), \qquad \text{for all} \quad s \in I \\ H(0,t) = H(1,t), \qquad \text{for all} \quad t \in I. \end{aligned} \tag{*}$$

- Put $\gamma_t(s) = H(s, t)$. Then (*) defines a one-parameter family of closed curves γ_t in X, which connects γ_0 and γ_1. Intuitively, this means that γ_0 can be continuously deformed to γ_1, within X.
- If γ_0 is X-homotopic to a constant mapping γ_1 (i.e., if γ_1^* consists of just one point), we say that γ_0 is **null-homotopic** in X.

Simply connected spaces

Definition

If X is connected and if every closed curve in X is null-homotopic, X is said to be **simply connected**.

Example

Every convex region Ω is simply connected. To see this, let γ_0 be a closed curve in Ω, fix $z_1 \in \Omega$, and define

$$H(s,t) = (1-t)\gamma_0(s) + tz_1 \quad \text{for all} \quad s,t \in [0,1].$$

Lemma

Let $\gamma_0, \gamma_1 : [0,1] \to \mathbb{C}$ be closed paths. Let $\alpha \in \mathbb{C}$, if

$$|\gamma_1(s) - \gamma_0(s)| < |\alpha - \gamma_0(s)| \quad \textit{for all} \quad s \in I = [0,1], \tag{X}$$

then $\operatorname{Ind}_{\gamma_1}(\alpha) = \operatorname{Ind}_{\gamma_0}(\alpha)$.

Homotopic curves

Proof: First we derive from (X) that $\alpha \notin \gamma_0^*$ and $\alpha \notin \gamma_1^*$.

- If $\alpha \in \gamma_0^*$, then $\alpha = \gamma_0(s)$ for some $s \in I$ and then by (X), we obtain

$$0 \leq |\gamma_1(s) - \gamma_0(s)| < |\gamma_0(s) - \gamma_0(s)| < 0.$$

  This is a contradiction.

- If $\alpha \in \gamma_1^*$, then $\alpha = \gamma_1(s)$ for some $s \in I$ and by (X), we obtain

$$|\gamma_1(s) - \gamma_0(s)| < |\gamma_1(s) - \gamma_0(s)|,$$

  which is again a contradiction.

- Now, since $\alpha \notin \gamma_0^*$, and $\alpha \notin \gamma_1^*$, then

$$\mathrm{Ind}_{\gamma_0}(\alpha) \quad \text{and} \quad \mathrm{Ind}_{\gamma_1}(\alpha)$$

  are defined.

Homotopic curves

- We consider
$$\gamma(s) = \frac{\gamma_1(s) - \alpha}{\gamma_0(s) - \alpha} \quad \text{for} \quad s \in I.$$
- We see that γ is a curve since $\alpha \notin \gamma_0^*$. Further, γ is a closed path since γ_0 and γ_1 are closed paths, and
$$\gamma'(s) = \frac{(\gamma_0(s) - \alpha)\,\gamma_1'(s) - (\gamma_1(s) - \alpha)\,\gamma_0'(s)}{(\gamma_0(s) - \alpha)^2}.$$
- We have
$$\gamma(s) - 1 = \frac{\gamma_1(s) - \alpha}{\gamma_0(s) - \alpha} - 1 = \frac{\gamma_1(s) - \gamma_0(s)}{\gamma_0(s) - \alpha}.$$
- Therefore by (X) we deduce
$$|\gamma(s) - 1| < 1 \quad \text{for} \quad s \in I.$$
- Thus $\gamma^* \subseteq D(1, 1)$. Therefore 0 lies in the unbounded region determined by γ and hence $\text{Ind}_\gamma(0) = 0$ by the index theorem.

Homotopic curves

- Now
$$0 = \mathrm{Ind}_{\gamma}(0) = \frac{1}{2\pi i}\int_{\gamma}\frac{dz}{z} = \frac{1}{2\pi i}\int_{0}^{1}\frac{\gamma'(s)}{\gamma(s)}ds$$
- By the computations from the previous slide
$$\begin{aligned}\frac{\gamma'(s)}{\gamma(s)} &= \frac{(\gamma_0(s)-\alpha)\,\gamma_1'(s) - (\gamma_1(s)-\alpha)\,\gamma_0'(s)}{(\gamma_0(s)-\alpha)\,(\gamma_1(s)-\alpha)} \\ &= \frac{\gamma_1'(s)}{\gamma_1(s)-\alpha} - \frac{\gamma_0'(s)}{\gamma_0(s)-\alpha}.\end{aligned}$$
- Therefore
$$\frac{1}{2\pi i}\int_{0}^{1}\frac{\gamma_1'(s)}{\gamma_1(s)-\alpha}ds = \frac{1}{2\pi i}\int_{0}^{1}\frac{\gamma_0'(s)}{\gamma_0(s)-\alpha}ds,$$
and hence
$$\mathrm{Ind}_{\gamma_0}(\alpha) = \mathrm{Ind}_{\gamma_1}(\alpha). \qquad \square$$

Homotopic curves

Theorem

If Γ_0 and Γ_1 are Ω-homotopic closed paths in a region $\Omega \subseteq \mathbb{C}$, and if $\alpha \notin \Omega$, then

$$\mathrm{Ind}_{\Gamma_1}(\alpha) = \mathrm{Ind}_{\Gamma_0}(\alpha).$$

Proof: By definition, there is a continuous $H : I^2 \to \Omega$ such that

$$H(s,0) = \Gamma_0(s), \quad H(s,1) = \Gamma_1(s), \quad \text{for all} \quad s \in I,$$
$$H(0,t) = H(1,t) \quad \text{for all} \quad t \in I.$$

- Since I^2 is compact, so is $H\left(I^2\right)$. Moreover, Ω^c is closed and $\Omega \cap H\left(I^2\right) = \emptyset$. Hence there exists $\varepsilon > 0$ such that

$$|\alpha - H(s,t)| > 2\varepsilon \quad \text{if} \quad (s,t) \in I^2.$$

Homotopic curves

- Since H is uniformly continuous, there is $n \in \mathbb{N}$ such that

$$\left|H(s,t) - H\left(s',t'\right)\right| < \varepsilon \quad \text{if} \quad \left|s-s'\right| + \left|t-t'\right| \leq 1/n$$

- Define polygonal closed paths $\gamma_0, \ldots, \gamma_n$ by

$$\gamma_k(s) = H\left(\frac{i}{n}, \frac{k}{n}\right)(ns+1-i) + H\left(\frac{i-1}{n}, \frac{k}{n}\right)(i-ns)$$

  if $i-1 \leq ns \leq i$ (equivalently $0 \leq i - ns \leq 1$), and $i = 1, \ldots, n$.
- Combining the last two inequalities, we obtain

$$\begin{aligned}|\gamma_k(s) - H(s,k/n)| \leq & |H(i/n,k/n) - H(s,k/n)|(ns+1-i) \\ & + |H((i-1)/n,k/n) - H(s,k/n)|(i-ns) < \varepsilon\end{aligned}$$

  for $k = 0, \ldots, n$ and $s \in I$.

Homotopic curves

- In particular, taking $k = 0$ and $k = n$, we obtain

$$|\gamma_0(s) - \Gamma_0(s)| < \varepsilon, \quad \text{and} \quad |\gamma_n(s) - \Gamma_1(s)| < \varepsilon.$$

- By the fact that $|\alpha - H(s,t)| > 2\varepsilon$ for all $(s,t) \in I^2$ and the last inequality from the previous slide $|\gamma_k(s) - H(s, k/n)| < \varepsilon$ for $s \in I$ and $k = 0, \ldots, n$, we obtain that

$$|\alpha - \gamma_k(s)| > \varepsilon \quad \text{for} \quad k = 0, \ldots, n \quad \text{and} \quad s \in I.$$

- Observe also that

$$\begin{aligned}|\gamma_{k-1}(s) - \gamma_k(s)| \leq |H\left(i/n, (k-1)/n\right) - H\left(i/n, k/n\right)|(ns + 1 - i) \\ + |H\left((i-1)/n, (k-1)/n\right) - H((i-1)/n, k/n)|(i - ns) < \varepsilon\end{aligned}$$

for $k = 0, \ldots, n$ and $s \in I$.

- Hence

$$|\gamma_{k-1}(s) - \gamma_k(s)| < |\alpha - \gamma_k(s)|$$

for $k = 0, \ldots, n$ and $s \in I$.

Homotopic curves

- Now by the previous lemma, observe that

$$\begin{aligned}
|\gamma_n(s) - \Gamma_1(s)| < |\alpha - \gamma_n(s)| &\implies \mathrm{Ind}_{\gamma_n}(\alpha) = \mathrm{Ind}_{\Gamma_1}(\alpha), \\
|\gamma_{n-1}(s) - \gamma_n(s)| < |\alpha - \gamma_n(s)| &\implies \mathrm{Ind}_{\gamma_{n-1}}(\alpha) = \mathrm{Ind}_{\gamma_n}(\alpha), \\
&\vdots \\
|\gamma_0(s) - \gamma_1(s)| < |\alpha - \gamma_1(s)| &\implies \mathrm{Ind}_{\gamma_0}(\alpha) = \mathrm{Ind}_{\gamma_1}(\alpha), \\
|\gamma_0(s) - \Gamma_0(s)| < |\alpha - \gamma_0(s)| &\implies \mathrm{Ind}_{\gamma_0}(\alpha) = \mathrm{Ind}_{\Gamma_0}(\alpha).
\end{aligned}$$

- This proves that

$$\mathrm{Ind}_{\Gamma_1}(\alpha) = \mathrm{Ind}_{\Gamma_0}(\alpha)$$

for every $\alpha \in \Omega^c$ as desired. □

Homotopic curves

Theorem

Suppose that $\Omega \subseteq \mathbb{C}$ is a simply connected region. Then for every closed curve Γ in Ω we have

$$\mathrm{Ind}_\Gamma(\alpha) = 0 \quad \textit{whenever} \quad \alpha \notin \Omega.$$

Proof: The region Ω is simply connected, hence every closed curve Γ in Ω is null-homotopic. In other words, there exists a continuous $H : I^2 \to \Omega$ such that

$$H(s,0) = \Gamma(s), \quad H(s,1) = \gamma(s), \quad \text{for all} \quad s \in I,$$
$$H(0,t) = H(1,t) \quad \text{for all} \quad t \in I,$$

where γ is a constant curve. Hence

$$\mathrm{Ind}_\gamma(\alpha) = 0 \quad \text{whenever} \quad \alpha \notin \Omega.$$

By the previous theorem we obtain that $\mathrm{Ind}_\Gamma(\alpha) = \mathrm{Ind}_\gamma(\alpha) = 0$ for $\alpha \notin \Omega$ as desired. □

Homotopic curves

Remark

- The previous theorem shows that the global Cauchy theorem holds in simply connected regions $\Omega \subseteq \mathbb{C}$, since

$$\mathrm{Ind}_\Gamma(\alpha) = 0, \quad \text{whenever} \quad \alpha \notin \Omega$$

for every closed path Γ in Ω.

- The last but one theorem shows that if Γ_0 and Γ_1 are Ω-homotopic closed paths in a region $\Omega \subseteq \mathbb{C}$, and if $\alpha \notin \Omega$, then

$$\mathrm{Ind}_{\Gamma_1}(\alpha) = \mathrm{Ind}_{\Gamma_0}(\alpha),$$

which combined with the second part of the global Cauchy theorem ensures that

$$\int_{\Gamma_0} f(z)dz = \int_{\Gamma_1} f(z)dz.$$