Lecture 7

Maximum principle and open mapping theorem
First glimpse at complex logarithms and winding number

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Orthogonality relations

Theorem

If

$$f(z) = \sum_{n=0}^{\infty} c_n (z-a)^n \quad \textit{for} \quad z \in D(a;R) \tag{*}$$

and if $0 < r < R$, *then*

$$\sum_{n=0}^{\infty} |c_n|^2 r^{2n} = \frac{1}{2\pi} \int_{-\pi}^{\pi} \left| f\left(a + re^{i\theta}\right) \right|^2 d\theta \tag{**}$$

Proof: We have

$$f\left(a + re^{i\theta}\right) = \sum_{n=0}^{\infty} c_n r^n e^{in\theta}.$$

For $r < R$, the series converges uniformly on $[-\pi, \pi]$.

Orthogonality relations

- Since

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} e^{-in\theta}\,d\theta = \begin{cases} 1 & \text{if } n = 0, \\ 0 & \text{if } n \in \mathbb{Z}\setminus\{0\}, \end{cases}$$

we conclude that

$$c_n r^n = \frac{1}{2\pi}\int_{-\pi}^{\pi} f\left(a + re^{i\theta}\right) e^{-in\theta}\,d\theta,$$

and

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} \left|f\left(a + re^{i\theta}\right)\right|^2 d\theta = \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} c_n\overline{c_m} r^{n+m}\frac{1}{2\pi}\int_{-\pi}^{\pi} e^{i(n-m)\theta}\,d\theta$$
$$= \sum_{n=0}^{\infty} \left|c_n\right|^2 r^{2n}.$$

- This completes the proof. □

Maximum modulus principle

Definition

Let f be defined on Ω and $a \in \Omega$. Then $|f|$ has a **local maximum** at a if there exists $\delta > 0$ such that $D(a, \delta) \subseteq \Omega$ and $|f(a)| \geq |f(z)|$ for every $z \in D(a, \delta)$. Further, we say that $|f|$ has no local maximum in Ω if $|f|$ does not have local maximum at every point of Ω. Similarly, we define a **local minimum**.

Theorem

Suppose that Ω is a region and $f \in H(\Omega)$.

(a) *Then $|f|$ has no local maximum at any point of Ω, unless f is constant.*

(b) *Moreover, if the closure of Ω is compact and f is continuous on $\overline{\Omega}$, then*

$$\sup_{z \in \Omega} |f(z)| \leq \sup_{z \in \partial\Omega} |f(z)|.$$

Maximum modulus principle

Proof: We now prove part (a). Suppose that $|f|$ attains a local maximum at $a \in \Omega$, then

$$|f(z)| \leq |f(a)| \quad \text{for all} \quad z \in D(a, \delta)$$

for some $\delta > 0$.

- If $z \in D(a, \delta)$, then it can be represented as

$$z = a + re^{i\theta}$$

  for some $r \in [0, \delta)$ and $\theta \in [-\pi, \pi]$. Hence

$$|f(a + re^{i\theta})| \leq |f(a)| \quad \text{for all} \quad r \in [0, \delta) \text{ and } \theta \in [-\pi, \pi].$$

- Since

$$f(z) = \sum_{n \geq 0} c_n (z - a)^n \quad \text{for} \quad z \in D(a, \delta).$$

Maximum modulus principle

- By the previous theorem it follows that

$$\sum_{n=0}^{\infty} |c_n|^2 r^{2n} = \frac{1}{2\pi}\int_{-\pi}^{\pi} \left|f\left(a + re^{i\theta}\right)\right|^2 d\theta \leq |f(a)|^2 = |c_0|^2 .$$

- Hence

$$\sum_{n=1}^{\infty} |c_n|^2 r^{2n} \leq 0$$

  and consequently $c_1 = c_2 = c_3 = \cdots = 0$, which implies that $f(z) = f(a)$ in $D(a; r)$.
- Since Ω is connected, then f must be constant in Ω.

We now prove part (b). There is nothing to prove if f is constant. Suppose that f is non-constant in Ω.

- Since f is continuous on $\overline{\Omega}$, which is compact, then it attains its maximum in $\overline{\Omega}$.
- Since f is non-constant this maximum must be attained in $\partial\Omega$, otherwise f would be constant by part (a). □

Maximum modulus principle

Corollary (Prove it!)

Suppose that Ω *is a region,* $f \in H(\Omega)$, *and* $\overline{D}(a; r) \subseteq \Omega$.

- *Then*

$$|f(a)| \leq \max_{\theta \in [-\pi, \pi]} \left| f\left(a + re^{i\theta}\right) \right|. \tag{*}$$

  Equality occurs in (*) if and only if f *is constant in* Ω. *Consequently,* $|f|$ *has no local maximum at any point of* Ω, *unless* f *is constant.*

- *Moreover, we have*

$$|f(a)| \geq \min_{\theta \in [-\pi, \pi]} \left| f\left(a + re^{i\theta}\right) \right|$$

  if f *has no zero in* $D(a; r)$.

Auxiliary lemma

Lemma

If $f \in H(\Omega)$ and g is defined in $\Omega \times \Omega$ by

$$g(z,w) = \begin{cases} \frac{f(z)-f(w)}{z-w} & \text{if } w \neq z, \\ f'(z) & \text{if } w = z, \end{cases}$$

then g is continuous in $\Omega \times \Omega$.

Proof: The only points $(z,w) \in \Omega \times \Omega$ at which the continuity of g is possibly in doubt have $z = w$.

Auxiliary lemma

- Fix $a \in \Omega$, and $\varepsilon > 0$. There exists $r > 0$ such that $D(a; r) \subseteq \Omega$ and
$$\left| f'(\zeta) - f'(a) \right| < \varepsilon$$
for all $\zeta \in D(a; r)$.
- If z and w are in $D(a; r)$ and if
$$\zeta(t) = (1 - t)z + tw,$$
then $\zeta(t) \in D(a; r)$ for $0 \leq t \leq 1$, and
$$g(z, w) - g(a, a) = \int_0^1 \left[f'(\zeta(t)) - f'(a) \right] dt.$$
- The absolute value of the integrand is $< \varepsilon$, for every $t \in [0, 1]$. Thus $|g(z, w) - g(a, a)| < \varepsilon$. This proves that g is continuous at (a, a). □

Holomorphic functions with nonvanishing derivatives

Theorem

Let $\Omega \subseteq \mathbb{C}$ be open. Suppose that $\varphi \in H(\Omega)$, and $\varphi'(z_0) \neq 0$ for some $z_0 \in \Omega$. Then Ω contains a neighborhood V of z_0 such that

(a) *φ is one-to-one in V;*

(b) *$W = \varphi(V)$ is an open set;*

(c) *if $\psi : W \to V$ is defined by $\psi(\varphi(z)) = z$, then $\psi \in H(W)$.*

Thus $\varphi : V \to W$ has a holomorphic inverse.

Proof: By the previous lemma applied to φ in place of f, we conclude that Ω contains a neighborhood V of z_0 such that

$$|\varphi(z_1) - \varphi(z_2)| \geq \frac{1}{2} \left|\varphi'(z_0)\right| |z_1 - z_2| \tag{1}$$

if $z_1 \in V$ and $z_2 \in V$. Thus (a) holds, and $\varphi'(z) \neq 0$ for $z \in V$.

Holomorphic function with nonvanishing derivatives

- To prove (b), pick $a \in V$ and choose $r > 0$ so that $\overline{D}(a, r) \subseteq V$.
- By inequality (1) there exists $c > 0$ such that

$$\left|\varphi\left(a + re^{i\theta}\right) - \varphi(a)\right| > 2c \quad \text{for} \quad \theta \in [-\pi, \pi]. \tag{2}$$

- If $\lambda \in D(\varphi(a); c)$, then $|\lambda - \varphi(a)| < c$, hence (2) implies

$$\min_{\theta \in [-\pi,\pi]} \left|\lambda - \varphi\left(a + re^{i\theta}\right)\right| > c > |\lambda - \varphi(a)|. \tag{3}$$

- By the previous corollary $\lambda - \varphi$ must therefore have a zero in $D(a; r)$.
- Thus $\lambda = \varphi(z)$ for some $z \in D(a; r) \subseteq V$.
- This proves that $D(\varphi(a); c) \subseteq \varphi(V)$.
- Hence $\varphi(V)$ is open, since a was an arbitrary point of V.

Holomorphic function with nonvanishing derivatives

- To prove (c), fix $w_1 \in W$.
- Then $\varphi(z_1) = w_1$ for a unique $z_1 \in V$ by property (b).
- If $w \in W$ and $\psi(w) = z \in V$, we have

$$\frac{\psi(w) - \psi(w_1)}{w - w_1} = \frac{z - z_1}{\varphi(z) - \varphi(z_1)}.$$

- By inequality (1) we deduce that $z \to z_1$ when $w \to w_1$.
- Moreover, $\phi'(z) \neq 0$ if $z \in V$, also by (1).
- Hence

$$\psi'(w_1) = \lim_{w \to w_1} \frac{\psi(w) - \psi(w_1)}{w - w_1} = \lim_{z \to z_1} \frac{z - z_1}{\varphi(z) - \varphi(z_1)} = 1/\varphi'(z_1).$$

- Thus $\psi \in H(W)$ as desired. □

Open mapping theorem

Definition

For $m \in \mathbb{N}$, we denote the m^{th}-power function $z \mapsto z^m$ by π_m.

- Each $w \neq 0$ is $\pi_m(z)$ for precisely m distinct values of z: If $w = re^{i\theta}$ for some $r > 0$, then

$$\pi_m(z) = w \quad \Longleftrightarrow \quad z = r^{1/m} e^{i(\theta + 2k\pi)/m} \quad \text{for} \quad k = 1, \ldots, m.$$

- Note also that each π_m is an open mapping: If V is open and does not contain 0, then $\pi_m(V)$ is open by the previous theorem. On the other hand, $\pi_m(D(0,r)) = D\left(0, r^m\right)$.
- Compositions of open mappings are clearly open. In particular, $\pi_m \circ \varphi$ is open, by the previous theorem, if φ' has no zero.

Open mapping theorem

Theorem

Suppose $\Omega \subseteq \mathbb{C}$ is open, $f \in H(\Omega)$ and f is not constant, $z_0 \in \Omega$, and $w_0 = f(z_0)$. Let m be the order of the zero which the function $f - w_0$ has at z_0. Then there exists a neighborhood $V \subseteq \Omega$ of z_0, and there exists $\varphi \in H(V)$, such that

(a) $f(z) = w_0 + [\varphi(z)]^m$ *for all* $z \in V$.

(b) *Moreover,* φ' *has no zero in* V *and* φ *is an invertible mapping of* V *onto a disc* $D(0; r)$.

Remark

Thus $f - w_0 = \pi_m \circ \varphi$ in V. It follows that f is an exactly m-to-1 mapping of $V \setminus \{z_0\}$ onto $D'(w_0; r^m)$, and that each $w_0 \in f(\Omega)$ is an interior point of $f(\Omega)$. Hence $f(\Omega)$ is open.

Open mapping theorem

Proof: Without loss of generality we may assume that Ω is a convex neighborhood of z_0 which is so small that $f(z) \neq w_0$ if $z \in \Omega \setminus \{z_0\}$.

- Then
$$f(z) - w_0 = (z - z_0)^m g(z) \quad \text{for} \quad z \in \Omega$$
for some $g \in H(\Omega)$ which has no zero in Ω. Hence $g'/g \in H(\Omega)$.
- By the Cauchy theorem $g'/g = h'$ for some $h \in H(\Omega)$.
- The derivative of $g \cdot \exp(-h)$ is 0 in Ω.
- If h is modified by the addition of a suitable constant, it follows that $g = \exp(h)$. Define
$$\varphi(z) = (z - z_0) \exp \frac{h(z)}{m} \quad \text{for} \quad z \in \Omega.$$
- Then (a) holds, for all $z \in \Omega$.
- Also, $\varphi(z_0) = 0$ and $\varphi'(z_0) \neq 0$. The existence of an open set V that satisfies (b) follows now from the previous theorem. □

Inverse mapping theorem

Theorem

Suppose that $\Omega \subseteq \mathbb{C}$ is open, $f \in H(\Omega)$, and f is one-to-one in Ω. Then $f'(z) \neq 0$ for every $z \in \Omega$, and the inverse of f is holomorphic.

Proof: If $f'(z_0)$ were 0 for some $z_0 \in \Omega$, the hypotheses of the previous theorem would hold with some $m > 1$, so that f would be m-to-1 in some deleted neighborhood of z_0, which is impossible, since f is one-to-one. Thus $f'(z) \neq 0$ for all $z \in \Omega$. Now apply part (c) of the last but one theorem. This completes the proof of the theorem. □

Remark

We observe that the converse of the inverse mapping theorem is false: If $f(z) = e^z$, then $f'(z) \neq 0$ for every $z \in \mathbb{C}$, but f is not one-to-one in the whole complex plane.

Complex logarithms and arguments

- The exponential function $S_\alpha \ni z \mapsto e^z \in \mathbb{C} \setminus \{0\}$ when restricted to the strip $S_\alpha = \{x + iy : \alpha \le y < \alpha + 2\pi\}$ is a one-to-one analytic map of this strip onto $\mathbb{C} \setminus \{0\}$ the nonzero complex numbers.

Definition

- We take $\log_\alpha$ to be the **inverse** of the exponential function restricted to the strip $S_\alpha = \{x + iy : \alpha \le y < \alpha + 2\pi\}$.
- We define $\arg_\alpha$ to the **imaginary part** of $\log_\alpha$.
- Consequently, $\log_\alpha(\exp z) = z$ for each $z \in S_\alpha$, and $\exp(\log_\alpha z) = z$ for all $z \in \mathbb{C} \setminus \{0\}$.

Definition

- The **principal branches** of the logarithm and argument functions, to be denoted by Log and Arg, are obtained by taking $\alpha = -\pi$.
- Thus, $\mathrm{Log} = \log_{-\pi}$ and $\mathrm{Arg} = \arg_{-\pi}$.

Complex logarithms and arguments

Theorem

(a) *If* $z \neq 0$, *then* $\log_\alpha(z) = \log|z| + i\arg_\alpha(z)$, *and* $\arg_\alpha(z)$ *is the unique number in* $[\alpha, \alpha + 2\pi)$ *such that*

$$z/|z| = e^{i\arg_\alpha(z)}.$$

In other words, the unique argument of z *in* $[\alpha, \alpha + 2\pi)$.

(b) *Let* $R_\alpha = \{re^{i\alpha} : r \geq 0\}$. *The functions* $\log_\alpha$ *and* $\arg_\alpha$ *are continuous at each point of the "slit" complex plane* $\mathbb{C} \backslash R_\alpha$, *and discontinuous at each point of* R_α.

(c) *The function* $\log_\alpha$ *is analytic on* $\mathbb{C} \backslash R_\alpha$, *and its derivative is given by* $\log'_\alpha(z) = 1/z$.

Complex logarithms and arguments

Proof:

(a) If $w = \log_\alpha(z)$ with $z \neq 0$, then $e^w = z$, hence

$$|z| = e^{\operatorname{Re} w}, \quad \text{and} \quad z/|z| = e^{i \operatorname{Im} w}.$$

Thus $\operatorname{Re} w = \log|z|$, and $\operatorname{Im} w$ is an argument of $z/|z|$. Since $\operatorname{Im} w$ is restricted to $[\alpha, \alpha + 2\pi)$ by definition of $\log_\alpha$, it follows that $\operatorname{Im} w$ is the unique argument for z that lies in the interval $[\alpha, \alpha + 2\pi)$.

(b) By (a), it suffices to consider $\arg_\alpha$. If $z_0 \in \mathbb{C} \backslash R_\alpha$ and $(z_n)_{n \in \mathbb{N}}$ is a sequence converging to z_0, then $\arg_\alpha(z_n)$ must converge to $\arg_\alpha(z_0)$. On the other hand, if $z_0 \in R_\alpha \backslash \{0\}$, there is a sequence $(z_n)_{n \in \mathbb{N}}$ converging to z_0 so that

$$\lim_{n \to \infty} \arg_\alpha(z_n) = \alpha + 2\pi \neq \arg_\alpha(z_0) = \alpha.$$

Continuous logarithms and arguments

Recall from Lecture 2 the following theorem:

Theorem

Let g be analytic on the open set Ω_1, and let f be a continuous complex-valued function on the open set Ω. Assume that

(i) $f(\Omega) \subseteq \Omega_1$,

(ii) g' *is never 0* ,

(iii) $g(f(z)) = z$ *for all* $z \in \Omega$ *(thus f is one-to-one).*

Then f is analytic on Ω and $f' = 1/\left(g' \circ f\right)$.

(c) By this theorem with $g = \exp$, $\Omega_1 = \mathbb{C}$, $f = \log_\alpha$, and $\Omega = \mathbb{C} \backslash R_\alpha$ and the fact that exp is its own derivative we obtain that

$$(\log_\alpha z)' = \frac{1}{z}.$$

This completes the proof of the theorem. □

Continuous logarithms and arguments

Definition

Let S be a subset of $\mathbb{C}$ (or more generally any metric space), and let $f : S \to \mathbb{C}\backslash\{0\}$ be continuous.

- A function $g : S \to \mathbb{C}$ is a continuous logarithm of f if g is continuous on S and $f(s) = e^{g(s)}$ for all $s \in S$.
- A function $\theta : S \to \mathbb{R}$ is a continuous argument of f if θ is continuous on S and $f(s) = |f(s)|e^{i\theta(s)}$ for all $s \in S$.

Examples

(a) If $S = [0, 2\pi]$ and $f(s) = e^{is}$, then f has a continuous argument on S, namely $\theta(s) = s + 2k\pi$ for any fixed integer k.

(b) If f is a continuous mapping of S into $\mathbb{C}\backslash R_\alpha$ for some $\alpha \in \mathbb{R}$, then f has a continuous argument, namely $\theta(s) = \arg_\alpha(f(s))$.

(c) If $S = \{z : |z| = 1\}$ and $f(z) = z$, then f does not have a continuous argument on S.

Continuous logarithms and arguments

Theorem

Let $f : S \to \mathbb{C}$ be continuous.

(a) *If g is a continuous logarithm of f, then $\operatorname{Im} g$ is a continuous argument of f.*

(b) *If θ is a continuous argument of f, then $\log|f| + i\theta$ is a continuous logarithm of f. Thus f has a continuous logarithm iff f has a continuous argument.*

(c) *Assume that S is connected, and f has continuous logarithms g_1 and g_2, and continuous arguments θ_1 and θ_2. Then there are integers k and l such that $g_1(s) - g_2(s) = 2\pi ik$ and $\theta_1(s) - \theta_2(s) = 2\pi l$ for all $s \in S$. Thus $g_1 - g_2$ and $\theta_1 - \theta_2$ are constant on S.*

(d) *If S is connected and $s, t \in S$, then*

$$g(s) - g(t) = \log|f(s)| - \log|f(t)| + i(\theta(s) - \theta(t))$$

for all continuous logarithms g and all continuous arguments θ of f.

Continuous logarithms and arguments

Proof:

(a) If $f(s) = e^{g(s)}$, then $|f(s)| = e^{\operatorname{Re} g(s)}$, hence $f(s)/|f(s)| = e^{i \operatorname{Im} g(s)}$ as required.

(b) If $f(s) = |f(s)|e^{i\theta(s)}$, then $f(s) = e^{\log|f(s)|+i\theta(s)}$, so $\log|f| + i\theta$ is a continuous logarithm of f.

(c) We have $f(s) = e^{g_1(s)} = e^{g_2(s)}$, hence $e^{g_1(s)-g_2(s)} = 1$, for all $s \in S$. Hence $g_1(s) - g_2(s) = 2\pi i k(s)$ for some integer-valued function $S \ni s \mapsto k(s) \in \mathbb{Z}$. Since g_1 and g_2 are continuous on S, so is k. But S is connected, so k is a constant function. A similar proof applies to any pair of continuous arguments of f.

(d) If θ is a continuous argument of f, then $\log|f| + i\theta$ is a continuous logarithm of f by part (b). Thus if g is any continuous logarithm of f, then $g = \log|f| + i\theta + 2\pi i k$ by (c).

The result follows. □

Continuous logarithms and arguments

Theorem

Let $\gamma : [a, b] \to \mathbb{C}\backslash\{0\}$ be a continuous curve and $0 \notin \gamma^*$. Then γ has a continuous argument, and consequently a continuous logarithm.

Proof: Let $\varepsilon = \inf\{|\gamma(t)| : t \in [a, b]\}$ be the distance from 0 to γ^*.

- Then $\varepsilon > 0$ because $0 \notin \gamma^*$ and γ^* is a closed set.
- By the uniform continuity of γ on $[a, b]$, there is a partition $a = t_0 < t_1 < \cdots < t_n = b$ of $[a, b]$ such that if $1 \leq j \leq n$ and $t \in [t_{j-1}, t_j]$, then $\gamma(t) \in D\left(\gamma\left(t_j\right), \varepsilon\right)$.
- Since $\gamma : [t_{j-1}, t_j] \to D\left(\gamma\left(t_j\right), \varepsilon\right)$ is a continuous function and $D\left(\gamma\left(t_j\right), \varepsilon\right) \subseteq \mathbb{C} \setminus R_{\alpha_j}$ for some $\alpha_j \in [0, 2\pi]$, then each $\gamma_{|[t_{j-1}, t_j]}$ has a continuous argument $\theta_j(t) = \arg_{\alpha_j}(\gamma_{|[t_{j-1}, t_j]}(t))$
- Since $\theta_j\left(t_j\right)$ and $\theta_{j+1}\left(t_j\right)$ differ by an integer multiple of 2π, we may (if necessary) redefine θ_{j+1} on $[t_j, t_{j+1}]$ so that the relation $\theta_j \cup \theta_{j+1}$ is a continuous argument of γ on $[t_{j-1}, t_{j+1}]$. Proceeding in this way, we obtain a continuous argument of γ on the entire interval $[a, b]$. $\square$

Continuous logarithms and arguments

Theorem

Let $\gamma : [a, b] \to \mathbb{C}$ be a closed curve. Fix $z_0 \notin \gamma^*$, and let θ be a continuous argument of $\gamma - z_0$, (θ exists by the previous theorem). Then $\theta(b) - \theta(a)$ is an integer multiple of 2π. Furthermore, if θ_1 is another continuous argument of $\gamma - z_0$, then $\theta_1(b) - \theta_1(a) = \theta(b) - \theta(a)$.

Proof: We know that $(\gamma(t) - z_0) / |\gamma(t) - z_0| = e^{i\theta(t)}$,for $t \in [a, b]$.

- Since γ is a closed curve, $\gamma(a) = \gamma(b)$, hence

$$1 = \frac{\gamma(b) - z_0}{|\gamma(b) - z_0|} \cdot \frac{|\gamma(a) - z_0|}{\gamma(a) - z_0} = e^{i(\theta(b)-\theta(a))}.$$

- Consequently, $\theta(b) - \theta(a)$ is an integer multiple of 2π. If θ_1 is another continuous argument of $\gamma - z_0$, then we have that $\theta_1 - \theta = 2\pi l$ for some $l \in \mathbb{Z}$. Thus $\theta_1(b) = \theta(b) + 2\pi l$ and $\theta_1(a) = \theta(a) + 2\pi l$, so $\theta_1(b) - \theta_1(a) = \theta(b) - \theta(a)$. □

Winding number

Definition

Let $\gamma : [a, b] \to \mathbb{C}$ be a closed curve. If $z_0 \notin \gamma^*$, let θ_{z_0} be a continuous argument of $\gamma - z_0$. The **winding number** of z_0 with respect to γ, is defined by

$$W(\gamma, z_0) = \frac{\theta_{z_0}(b) - \theta_{z_0}(a)}{2\pi}.$$

Remarks

- By the previous theorem, $W(\gamma, z_0)$ is well-defined, that is, $W(\gamma, z_0)$ does not depend on the particular continuous argument chosen.
- Intuitively, $W(\gamma, z_0)$ is the net number of revolutions of $\gamma(t)$ about the point z_0 when $t \in [a, b]$.
- Note that by the above definition, for any complex number w we have $W(\gamma, z_0) = W(\gamma + w, z_0 + w)$.

Winding number

Theorem

Let $\gamma : [a, b] \to \mathbb{C}$ be a closed path, and z_0 a point not belonging to γ^*. Then

$$W(\gamma, z_0) = \frac{1}{2\pi i} \int_\gamma \frac{1}{z - z_0} dz = \mathrm{Ind}_\gamma(z_0).$$

In other words, the winding number is the index function. More generally, if f is analytic on an open set Ω containing γ^*, and $z_0 \notin (f \circ \gamma)^*$, then

$$W(f \circ \gamma, z_0) = \frac{1}{2\pi i} \int_\gamma \frac{f'(z)}{f(z) - z_0} dz.$$

Winding number

Proof: Let ε be the distance from z_0 to γ^*.

- Since γ is uniformly continuous on $[a, b]$, then there is a partition $a = t_0 < t_1 < \cdots < t_n = b$ so that

$$\gamma(t) \in D\left(\gamma\left(t_j\right), \varepsilon\right) \quad \text{for} \quad t \in [t_{j-1}, t_j].$$

- By definition of ε we have $z_0 \notin D\left(\gamma\left(t_j\right), \varepsilon\right)$ for each $1 \leq j \leq n$.
- Consequently, the holomorphic function $h(z) = z - z_0$, when restricted to $D\left(\gamma\left(t_j\right), \varepsilon\right)$ is nonvanishing.
- Therefore, $\frac{h'(z)}{h(z)} = \frac{1}{(z-z_0)}$ is holomorphic in $D\left(\gamma\left(t_j\right), \varepsilon\right)$.
- By the Cauchy theorem h'/h has a primitive g_j in $D\left(\gamma\left(t_j\right), \varepsilon\right)$.
- Therefore $g_j'(z) = 1/\left(z - z_0\right)$ for all $z \in D\left(\gamma\left(t_j\right), \varepsilon\right)$.
- The path γ restricted to $[t_{j-1}, t_j]$ lies in the disk $D\left(\gamma\left(t_j\right), \varepsilon\right)$, hence

$$\int_{\gamma_{|[t_{j-1}, t_j]}} \frac{1}{z - z_0} dz = g_j\left(\gamma\left(t_j\right)\right) - g_j\left(\gamma\left(t_{j-1}\right)\right).$$

Winding number

- Thus
$$\int_\gamma \frac{1}{z-z_0}dz = \sum_{j=1}^{n}\left[g_j\left(\gamma\left(t_j\right)\right) - g_j\left(\gamma\left(t_{j-1}\right)\right)\right].$$
- Observe that the function $\exp(-g_j)h$ has a vanishing derivative on $D\left(\gamma\left(t_j\right),\varepsilon\right)$. If g_j is modified by the addition of a suitable constant, it follows that $h=\exp(g_j)$. If $\theta_j = \operatorname{Im} g_j$, then θ_j is a continuous argument of $h(z)=z-z_0$ on $D\left(\gamma\left(t_j\right),\varepsilon\right)$.
- Since $g_j(z)=\log|z|+i\theta_j(z)$, then
$$\int_\gamma \frac{1}{z-z_0}dz = i\sum_{j=1}^{n}\left[\theta_j\left(\gamma\left(t_j\right)\right) - \theta_j\left(\gamma\left(t_{j-1}\right)\right)\right].$$
- If θ is any continuous argument of $\gamma - z_0$, then $\theta_{|[t_{j-1},t_j]}$ is a continuous argument of $(\gamma - z_0)_{|[t_{j-1},t_j]}$. But so is $\theta_j \circ \gamma_{|[t_{j-1},t_j]}$, hence
$$\theta_j\left(\gamma\left(t_j\right)\right) - \theta_j\left(\gamma\left(t_{j-1}\right)\right) = \theta\left(t_j\right) - \theta\left(t_{j-1}\right),$$
since two continuous arguments differs by a multiple of 2π.

Winding number

- Therefore,

$$\int_{\gamma} \frac{1}{z - z_0} dz = i \sum_{j=1}^{n} \left[\theta\left(t_j\right) - \theta\left(t_{j-1}\right)\right]$$
$$= i(\theta(b) - \theta(a))$$
$$= 2\pi i W\left(\gamma, z_0\right)$$

completing the proof of the first part of the theorem.

- Applying this result to the path $f \circ \gamma$, we get the second statement. Specifically, if $z_0 \notin (f \circ \gamma)^*$, then

$$W\left(f \circ \gamma, z_0\right) = \frac{1}{2\pi i} \int_{f \circ \gamma} \frac{1}{z - z_0} dz$$
$$= \frac{1}{2\pi i} \int_{\gamma} \frac{f'(z)}{f(z) - z_0} dz. \qquad \square$$