# Lecture 6

## Further applications of the Cauchy integral formula
## Singular points and their classification

MATH 503, FALL 2025

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# Cauchy inequality

### Corollary

*If $f$ is holomorphic in an open set $\Omega \subseteq \mathbb{C}$ that contains a closed disc $\overline{D}(z_0, R)$ centered at $z_0$ and of radius $R$, then*

$$\left| f^{(n)}(z_0) \right| \leq \frac{n! \|f\|_{L^\infty(C)}}{R^n},$$

*where $\|f\|_{L^\infty(C)} = \sup_{z \in C} |f(z)|$ denotes the supremum of $|f|$ on the boundary circle $C = \partial \overline{D}(z_0, R)$.*

# Liouville's theorem and fundamental theorem of algebra

**Theorem**

*If $f$ is entire and bounded, then $f$ is constant.*

**Proof:** Let $B = \sup_{z\in\mathbb{C}} |f(z)|$. For each $z_0 \in \mathbb{C}$ and all $R > 0$, the Cauchy inequality yields

$$\left|f'(z_0)\right| \leq \frac{B}{R} \xrightarrow[R\to\infty]{} 0.$$

Thus $f'(z_0) = 0$ and hence $f$ must be constant as $\mathbb{C}$ is connected. □

**Theorem**

*Every non-constant polynomial*

$$P(z) = a_n z^n + \cdots + a_1 z + a_0$$

*with complex coefficients $a_n, \ldots, a_0$ has a root in $\mathbb{C}$. In particular, if* $\deg P = n \geq 1$*, then it has precisely $n$ roots.*

# Fundamental theorem of algebra

**Proof:** If $P$ has no roots, then $1/P(z)$ is a bounded holomorphic function.

- To see this, we can of course assume that $a_n \neq 0$, and write

$$\frac{P(z)}{z^n} = a_n + \left(\frac{a_{n-1}}{z} + \cdots + \frac{a_0}{z^n}\right) \quad \text{whenever} \quad z \neq 0.$$

- Since each term in the parentheses goes to 0 as $|z| \to \infty$ we conclude that there exists $R > 0$ so that if $c = |a_n|/2$, then

$$|P(z)| \geq c|z|^n \quad \text{whenever} \quad |z| > R.$$

- In particular, $P$ is bounded from below when $|z| > R$.
- Since $P$ is continuous and has no roots in the disc $|z| \leq R$, it is bounded from below in that disc as well, thereby proving our claim.
- By Liouville's theorem we then conclude that $1/P$ is constant.
- This contradicts our assumption that $P$ is non-constant and proves the theorem. □

# Morera's theorem — converse to the Cauchy theorem

**Theorem**

*Suppose $f$ is a continuous complex function in an open set $\Omega \subseteq \mathbb{C}$ such that*

$$\int_{\partial\Delta} f(z)dz = 0$$

*for every closed triangle $\Delta \subset \Omega$. Then $f \in H(\Omega)$.*

**Proof:** Let $V$ be a convex open set in $\Omega$.

- As in the proof of Cauchy theorem for convex sets, we can construct a primitive $F \in H(V)$ such that $F' = f$.
- Fixing $a \in V$ it suffices to take

$$F(z) = \int_{[a,z]} f(\zeta)d\zeta \quad \text{for} \quad z \in V.$$

# Morera's theorem — converse to the Cauchy theorem

- Observe that $[a, z] \subseteq V$ since $V$ is convex. Let $z_0 \in V$, then

$$F(z) - F(z_0) = \int_{[a,z]} f(\zeta) d\zeta - \int_{[a,z_0]} f(\zeta) d\zeta$$
$$= \int_{[z_0,z]} f(\zeta) d\zeta.$$

- Hence, we obtain

$$\frac{F(z) - F(z_0)}{z - z_0} - f(z_0) = \frac{1}{z - z_0} \int_{[z_0,z]} (f(\zeta) - f(z_0))\, d\zeta,$$

which implies

$$F'(z_0) = \lim_{z \to z_0} \frac{F(z) - F(z_0)}{z - z_0} = f(z_0).$$

- Since derivatives of holomorphic functions are holomorphic, we have $f \in H(V)$, for every convex open $V \subseteq \Omega$, hence $f \in H(\Omega)$. □

# Sequences of holomorphic functions

**Definition**

A sequence $(f_j)_{j\in\mathbb{N}}$ of functions in $\Omega$ is said to converge to $f$ uniformly on compact subsets of $\Omega$ if to every compact $K \subseteq \Omega$ and to every $\varepsilon > 0$ there is $N = N(K, \varepsilon) \in \mathbb{N}$ such that $\sup_{z\in K} |f_j(z) - f(z)| < \varepsilon$ if $j > N$.

**Theorem**

*Let $\Omega \subseteq \mathbb{C}$ be open. Suppose $(f_j)_{j\in\mathbb{N}} \subseteq H(\Omega)$, and $\lim_{j\to\infty} f_j = f$ uniformly on compact subsets of $\Omega$. Then $f \in H(\Omega)$, and $\lim_{j\to\infty} f_j' = f'$ uniformly on compact subsets of $\Omega$.*

**Proof:** The function $f$ is continuous, since the convergence is uniform on each compact disc in $\Omega$. Let $\Delta$ be a triangle in $\Omega$. Then $\Delta$ is compact, so

$$\int_{\partial\Delta} f(z)dz = \lim_{j\to\infty} \int_{\partial\Delta} f_j(z)dz = 0$$

by Cauchy's theorem. Hence Morera's theorem implies that $f \in H(\Omega)$.

# Sequences of holomorphic functions

- Let $K \subseteq \Omega$ be compact. There exists an $r > 0$ such that the union $E$ of the closed discs $\overline{D}(z; r)$, for all $z \in K$, is a compact subset of $\Omega$.
- Applying Cauchy's inequality to $f - f_j$, we have

$$\left|f'(z) - f_j'(z)\right| \leq r^{-1} \left\|f - f_j\right\|_{L^\infty(E)} \quad \text{for} \quad z \in K,$$

where $\|f\|_{L^\infty(E)} = \sup_{z \in E} |f(z)|$. Since $\lim_{j\to\infty} f_j = f$ uniformly on $E$, it follows that $f_j' \to f'$ uniformly on $K$.

**Corollary**

*Under the same hypothesis,* $\lim_{j\to\infty} f_j^{(n)} = f^{(n)}$ *uniformly, on every compact set* $K \subseteq \Omega$*, and for every positive integer* $n \in \mathbb{N}$*.*

- Compare this with the situation on the real line, where sequences of infinitely differentiable functions can converge uniformly to nowhere differentiable functions!

# Zero sets of holomorphic functions

## Theorem

*Suppose that* $\Omega \subseteq \mathbb{C}$ *is a region, and* $f \in H(\Omega)$*, and*

$$Z(f) = \{a \in \Omega : f(a) = 0\}.$$

(i) *Then either* $Z(f) = \Omega$*, or*

(ii) $Z(f)$ *has no limit point in* $\Omega$*.*

*In the latter case there corresponds to each* $a \in Z(f)$ *a unique positive integer* $m = m(a)$ *such that*

$$f(z) = (z - a)^m g(z) \quad \text{for} \quad z \in \Omega, \tag{*}$$

*where* $g \in H(\Omega)$ *and* $g(a) \neq 0$*. Furthermore,* $Z(f)$ *is at most countable. The integer* $m$ *is called the* **order of the zero** *which* $f$ *has at the point* $a$*.*

# Zero sets of holomorphic functions

**Proof:** Let $A$ be the set of all limit points of $Z(f)$ in $\Omega$. In other words,

$$A = \{z \in \mathbb{C} : \exists_{(z_n)_{n\in\mathbb{N}} \subseteq Z(f)}\ z_n \neq z \ \text{ and } \lim_{n\to\infty} z_n = z\}.$$

- Since $f$ is continuous, then $A \subseteq Z(f)$. Indeed, if $z \in A$ then there $(z_n)_{n\in\mathbb{N}} \subseteq Z(f)$ such that $z_n \neq z$ and $\lim_{n\to\infty} z_n = z$. By continuity of $f$ we obtain that $f(z) = \lim_{n\to\infty} f(z_n) = 0$.
- We will show that $A$ is closed in $\Omega$. Indeed, if $(z_n)_{n\in\mathbb{N}} \subseteq A$ and $\lim_{n\to\infty} z_n = z_0$, then for any $\varepsilon > 0$ there exists $N_\varepsilon \in \mathbb{N}$ such that $z_n \in D(z_0, \varepsilon)$ for all $n \geq N_\varepsilon$. Since $z_n$ is a limit point of $Z(f)$, then $D(z_0, \varepsilon)$ contains infinitely many points of $Z(f)$ different from $z_n$, and hence infinitely many points of $Z(f)$ different from $z_0$. Thus $z_0 \in A$.
- This shows that $A$ is closed.

# Zero sets of holomorphic functions

- Fix $a \in Z(f)$, and choose $r > 0$ so that $D(a;r) \subseteq \Omega$, then

$$f(z) = \sum_{n=0}^{\infty} c_n (z-a)^n \quad \text{for} \quad z \in D(a;r).$$

- There are now two possibilities:
  - (i) Either $c_n = 0$ for all integers $n \geq 0$;
  - (ii) Or there is a smallest integer $m$ (necessarily positive, since $f(a) = 0 = c_0$) such that $c_m \neq 0$.
- In that case, we can define

$$g(z) = \begin{cases} (z-a)^{-m} f(z) & \text{if } z \in \Omega \setminus \{a\}, \\ c_m & \text{if } z = a. \end{cases}$$

- It is clear that $g \in H(\Omega \setminus \{a\})$.

# Zero sets of holomorphic functions

- Moreover, for $z \in D(a;r)$, we have

$$f(z) = (z-a)^m g(z) = (z-a)^m \Big( c_m + \sum_{n=1}^{\infty} c_{n+m}(z-a)^n \Big), \qquad (**)$$

  where

$$g(z) = \sum_{n=0}^{\infty} c_{n+m}(z-a)^n \quad \text{for} \quad z \in D(a;r).$$

- Since $g \in H(D(a;r))$, we consequently conclude that $g \in H(\Omega)$.
- Hence we obtain the factorization from (*) for some $g \in H(\Omega)$, i.e.

$$f(z) = (z-a)^m g(z) \quad \text{for} \quad z \in \Omega.$$

- Moreover, $g(a) = c_m \neq 0$, and the continuity of $g$ shows that there is a neighborhood of $a$ in which $g$ has no zero, by (**).
- Thus we have shown that $a$ is an isolated point of $Z(f)$.

# Zero sets of holomorphic functions

- If $a \in A$, and $a$ is a limit point of $Z(f)$ then the first case (i) must occur and all coefficients $c_n$ are 0, then $f(z) = 0$ for all $z \in D(a; r)$, and consequently $D(a; r) \subseteq A$ and $a$ is an interior point of $A$.
- This proves that $A$ is open.
- If $B = \Omega \setminus A$, then $B$ is open.
- Thus $\Omega$ is the union of the disjoint open sets $A$ and $B$.
- Since $\Omega$ is connected, we have

  (a) either $A = \Omega$, in which case $Z(f) = \Omega$,

  (b) or $A = \varnothing$.

- In the latter case, $Z(f)$ has at most finitely many points in each compact subset of $\Omega$, and since $\Omega$ is $\sigma$-compact, $Z(f)$ is at most countable. □

# Identity theorem

**Theorem**

*If $f$ and $g$ are holomorphic functions in a region $\Omega$ and if $f(z) = g(z)$ for all $z$ in some set which has a limit point in $\Omega$, then*

$$f(z) = g(z) \quad \textit{for all} \quad z \in \Omega.$$

**Definition**

Suppose we are given a pair of functions $f$ and $F$ analytic in regions $\Omega$ and $\Omega'$, respectively, with $\Omega \subseteq \Omega'$. If the two functions agree on the smaller set $\Omega$, we say that $F$ is an **analytic continuation** of $f$ into the region $\Omega'$.

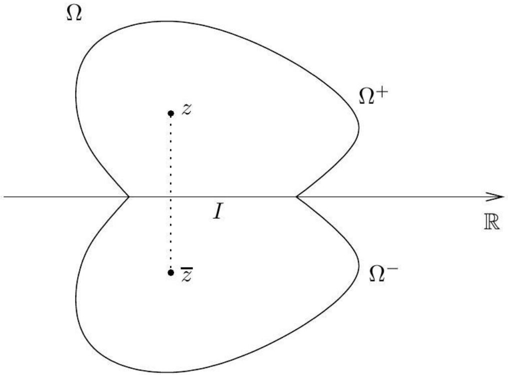
- The identity theorem then guarantees that there can be only one such analytic continuation, since $F$ is uniquely determined by $f$.

# Symmetry principle

Let $\Omega$ be an open subset of $\mathbb{C}$ that is symmetric with respect to the real line, that is

$$z \in \Omega \quad \text{if and only if} \quad \bar{z} \in \Omega.$$

Let $\Omega^+$denote the part of $\Omega$ that lies in the upper half-plane and $\Omega^-$ that part that lies in the lower half-plane.



# Symmetry principle

Also, let $I = \Omega \cap \mathbb{R}$ so that $I$ denotes the interior of that part of the boundary of $\Omega^+$ and $\Omega^-$ that lies on the real axis. Then we have

$$\Omega^+ \cup I \cup \Omega^- = \Omega.$$

**Theorem**

*If $f^+$ and $f^-$ are holomorphic functions in $\Omega^+$ and $\Omega^-$ respectively, that extend continuously to $I$ and*

$$f^+(x) = f^-(x) \quad \textit{for all} \quad x \in I,$$

*then the function $f$ defined on $\Omega$ by*

$$f(z) = \begin{cases} f^+(z) & \textit{if } z \in \Omega^+, \\ f^+(z) = f^-(z) & \textit{if } z \in I, \\ f^-(z) & \textit{if } z \in \Omega^-, \end{cases}$$

*is holomorphic on all of $\Omega$.*

# Symmetry principle

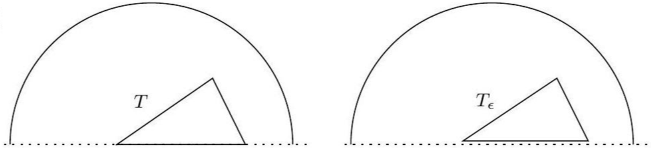
**Proof:** One notes first that $f$ is continuous throughout $\Omega$. The only difficulty is to prove that $f$ is holomorphic at points of $I$. Suppose $D$ is a disc centered at a point on $I$ and entirely contained in $\Omega$. We prove that $f$ is holomorphic in $D$ by Morera's theorem.

- Suppose $T$ is a triangle in $D$. If $T$ does not intersect $I$, then

$$\int_T f(z)dz = 0,$$

  since $f$ is holomorphic in the upper and lower half-discs.

- Suppose now that one side or vertex of $T$ is contained in $I$, and the rest of $T$ is in, say, the upper half-disc.

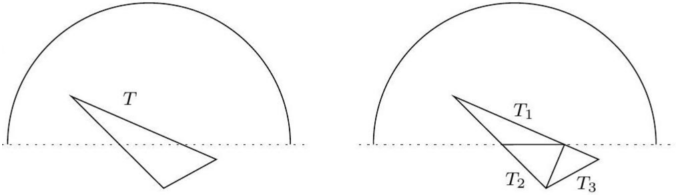


# Symmetry principle

- If $T_\epsilon$ is the triangle obtained from $T$ by slightly raising the edge or vertex which lies on $I$, we have $\int_{T_\epsilon} f(z)dz = 0$, since $T_\epsilon$ is entirely contained in the upper half-disc. We then let $\epsilon \to 0$, and by continuity we conclude that

$$\int_T f(z)dz = 0.$$

- If the interior of $T$ intersects $I$, like below




we can reduce the situation to the previous one by writing $T$ as the union of triangles each of which has an edge or vertex on $I$. By Morera's theorem $f$ is holomorphic in $D$, as desired. □

# Schwarz reflection principle

**Theorem**

*Suppose that $f$ is a holomorphic function in $\Omega^+$ that extends continuously to $I$ and such that $f$ is real-valued on $I$. Then there exists a function $F$ holomorphic in all of $\Omega$ such that $F = f$ on $\Omega^+$.*

Proof. The idea is simply to define $F(z)$ for $z \in \Omega^-$ by $F(z) = \overline{f(\overline{z})}$

- To prove that $F$ is holomorphic in $\Omega^-$ we note that if $z, z_0 \in \Omega^-$, then $\overline{z}, \overline{z_0} \in \Omega^+$ and hence, the power series expansion of $f$ near $\overline{z_0}$ gives

$$f(\overline{z}) = \sum a_n (\overline{z} - \overline{z_0})^n .$$

- As a consequence we see that $F$ is holomorphic in $\Omega^-$, since

$$F(z) = \sum \overline{a_n} (z - z_0)^n .$$

- Since $f$ is real valued on $I$ we have $\overline{f(x)} = f(x)$ whenever $x \in I$ and hence $F$ extends continuously up to $I$.
- The proof is complete once we invoke the symmetry principle. □

# Riemann's theorem on removable singularities

**Definition**

- If $a \in \Omega$ and $f \in H(\Omega \setminus \{a\})$, then $f$ is said to have an **isolated singularity** at the point $a$.
- If $f$ can be so defined at $a$ that the extended function is holomorphic in $\Omega$, the singularity is said to be **removable**.

**Theorem**

*Suppose $f \in H(\Omega \setminus \{a\})$ and $f$ is bounded in $D'(a;r)$, for some $r > 0$. Then $f$ has a removable singularity at $a$.*

**Proof:** Define

$$h(z) = \begin{cases} (z-a)^2 f(z) & \text{if } z \in \Omega \setminus \{a\}, \\ 0 & \text{if } z = a. \end{cases}$$

# Riemann's theorem on removable singularities

- Our boundedness assumption shows that

$$h'(a) = \lim_{z \to a} \frac{h(z) - h(a)}{z - a} = \lim_{z \to a} (z - a) f(z) = 0.$$

- Since $h$ is evidently differentiable at every other point of $\Omega$, we have $h \in H(\Omega)$. Moreover, $h(a) = h'(a) = 0$, so

$$h(z) = \sum_{n=2}^{\infty} c_n (z - a)^n \quad \text{for} \quad z \in D(a; r).$$

- We obtain the desired holomorphic extension of $f$ by setting $f(a) = c_2$, for then

$$f(z) = \sum_{n=0}^{\infty} c_{n+2} (z - a)^n \quad \text{for} \quad z \in D(a; r). \qquad \square$$

# Singularities

## Theorem

*If $a \in \Omega$ and $f \in H(\Omega \setminus \{a\})$, then $f$ has a singularity at $a$, and one of the following three cases must occur:*

(a) *$f$ has a removable singularity at $a$.*

(b) *There are $c_1, \ldots, c_m \in \mathbb{C}$ for some $m \in \mathbb{N}$ and $c_m \neq 0$, such that*

$$f(z) - \sum_{k=1}^{m} \frac{c_k}{(z-a)^k}$$

*has a removable singularity at $a$.*

(c) *If $r > 0$ and $D(a; r) \subseteq \Omega$, then $f(D'(a; r))$ is dense in $\mathbb{C}$.*

- *The conclusion of item (iii) is called the Casorati–Weierstrass theorem.*

# Singularities

## Remarks

- In case $(b)$, $f$ is said to have a **pole of order** $m$ at $a$. The function

$$\sum_{k=1}^{m} c_k (z-a)^{-k}$$

  a polynomial in $(z-a)^{-1}$, is called the **principal part** of $f$ at $a$.
- It is clear in this situation that $|f(z)| \to \infty$ as $z \to a$.
- In case (c), $f$ is said to have an **essential singularity** at $a$.
- A statement equivalent to $(c)$ is that to each complex number $w \in \mathbb{C}$ there corresponds a sequence $(z_n)_{n\in\mathbb{N}}$ such that $\lim_{n\to\infty} z_n = a$ and

$$\lim_{n\to\infty} f(z_n) = w.$$

# Singularities

**Proof:** Suppose (c) fails.

- Then there exist $r > 0$, and $\delta > 0$, and $w \in \mathbb{C}$ such that

$$|f(z) - w| > \delta \quad \text{for all} \quad z \in D'(a; r).$$

- Let us write $D$ for $D(a; r)$ and $D'$ for $D'(a; r)$. Define

$$g(z) = \frac{1}{f(z) - w} \quad \text{for} \quad z \in D'. \tag{1}$$

- Then $g \in H(D')$ and $|g| < 1/\delta$. Since $g$ is bounded, then by the previous theorem it extends to a holomorphic function in $D$.
- If $g(a) \neq 0$, then formula (1) shows that $f$ is bounded in $D'(a; \rho)$ for some $\rho > 0$.
- Hence, by the previous theorem $f$ can be extended to a holomorphic function in $D(a; \rho)$, and consequently holomorphic in $\Omega$, yielding (a).

# Singularities

- If $g$ has a zero of order $m \geq 1$ at $a$, then we can write

$$g(z) = (z-a)^m g_1(z) \quad \text{for} \quad z \in D, \tag{2}$$

  where $g_1 \in H(D)$ and $g_1(a) \neq 0$. Also, $g_1$ has no zero in $D'$, by (1).
- Thus we can set $h = 1/g_1$ in $D$.
- Then $h \in H(D)$, and $h$ has no zero in $D$, and we can write

$$f(z) - w = (z-a)^{-m} h(z) \quad \text{for} \quad z \in D'. \tag{3}$$

- But $h$ has an expansion of the form

$$h(z) = \sum_{n=0}^{\infty} b_n (z-a)^n \quad \text{for} \quad z \in D, \tag{4}$$

  with $b_0 \neq 0$.
- Identity (3) shows that (b) holds, with $c_k = b_{m-k}$ for $k = 1, \ldots, m$.
- This completes the proof. □

# Remarks

Suppose $f$ has an isolated singularity at $z_0$, and let

$$\sum_{n=-\infty}^{\infty} a_n (z - z_0)^n$$

be the Laurent expansion of $f$ about $z_0$. Then one can see that

- $f$ has a **removable singularity** at $z_0$ if $a_n = 0$ for all $n < 0$.
- $f$ has a **pole of order** $m$ at $z_0$ if $m$ is the largest positive integer such that $a_{-m} \neq 0$. (A pole of order 1 is called a **simple pole**.)
- Finally, if $a_n \neq 0$ for infinitely many $n < 0$, we say that $f$ has an **essential singularity** at $z_0$.
- The behavior of a complex function $f$ at $\infty$ may be studied by considering $g(z) = f(1/z)$ for $z$ near 0. Then we say that $f$ has an **isolated singularity** at $\infty$ if $f$ is analytic on $\{z \in \mathbb{C} : |z| > r\}$ for some $r$; thus the function $g(z) = f(1/z)$ has an isolated singularity at 0. The type of singularity of $f$ at $\infty$ is then defined as that of $g$ at 0.