Lecture 5

Cauchy integral formula and its applications

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Cauchy integral formula

Recall that

Theorem (Cauchy theorem for convex sets)

Let $\Omega \subseteq \mathbb{C}$ be a convex open set and $p \in \Omega$. Let f be continuous in Ω and holomorphic in $\Omega \backslash \{p\}$. Then f has a primitive in Ω and

$$\int_\gamma f(z)dz = 0$$

for any closed path γ in Ω.

Theorem (Cauchy integral formula for convex sets)

Let γ be a closed path in an open convex set $\Omega \subseteq \mathbb{C}$ and $f \in H(\Omega)$. Then for any $a \in \Omega$ and $a \notin \gamma^*$, we have

$$\frac{1}{2\pi i}\int_\gamma \frac{f(z)}{z-a}dz = \mathrm{Ind}_\gamma(a)f(a).$$

Cauchy integral formula

Proof: We define for $z \in \Omega$ the following function

$$g(z) = \begin{cases} \frac{f(z)-f(a)}{z-a} & \text{if } z \neq a \\ f'(a) & \text{if } z = a \end{cases}$$

- We observe that g is continuous in Ω and $g \in H(\Omega \backslash \{a\})$.
- Now we apply the Cauchy theorem for convex sets (with $f = g$ and $p = a$), to conclude that

$$\int_\gamma g(z) dz = 0.$$

- Then, since $a \notin \gamma^*$, we have

$$\frac{1}{2\pi i} \int_\gamma \frac{f(z)}{z-a} dz = \frac{f(a)}{2\pi i} \int_\gamma \frac{dz}{z-a} = \text{Ind}_\gamma(a) f(a). \qquad \square$$

If $f \in H(\Omega)$, then $f^{(n)} \in H(\Omega)$ for any $n \in \mathbb{N}$

Corollary

If f is holomorphic in an open set $\Omega \subseteq \mathbb{C}$, then f has infinitely many complex derivatives in Ω. In other words,

- *If $f \in H(\Omega)$, then $f^{(n)} \in H(\Omega)$ for any $n \in \mathbb{N}$.*
- *Moreover, if $C \subset \Omega$ is a circle whose interior is also contained in Ω, then*

$$f^{(n)}(z) = \frac{n!}{2\pi i} \int_C \frac{f(\zeta) d\zeta}{(\zeta - z)^{n+1}}$$

for all z in the interior of C.

Proof: The proof is by induction on $n \geq 0$, the case $n = 0$ being simply the Cauchy integral formula.

- Suppose that f has up to $n-1$ complex derivatives and that

$$f^{(n-1)}(z) = \frac{(n-1)!}{2\pi i} \int_C \frac{f(\zeta)}{(\zeta - z)^n d\zeta}.$$

If $f \in H(\Omega)$, then $f^{(n)} \in H(\Omega)$ for any $n \in \mathbb{N}$

- Now for h small, the difference quotient for $f^{(n-1)}$ takes the form

$$\frac{f^{(n-1)}(z+h) - f^{(n-1)}(z)}{h} =$$
$$\frac{(n-1)!}{2\pi i} \int_C f(\zeta) \frac{1}{h} \left[\frac{1}{(\zeta - z - h)^n} - \frac{1}{(\zeta - z)^n} \right] d\zeta$$

- We now recall that

$$A^n - B^n = (A - B) \left[A^{n-1} + A^{n-2}B + \cdots + AB^{n-2} + B^{n-1} \right].$$

- With $A = 1/(\zeta - z - h)$ and $B = 1/(\zeta - z)$, we see that the term in brackets under the integral is equal to

$$\frac{h}{(\zeta - z - h)(\zeta - z)} \left[A^{n-1} + A^{n-2}B + \cdots + AB^{n-2} + B^{n-1} \right].$$

If $f \in H(\Omega)$, then $f^{(n)} \in H(\Omega)$ for any $n \in \mathbb{N}$

- Observe that if h is small, then $z + h$ and z stay at a finite distance from the boundary circle C, so in the limit as h tends to 0, we find that the quotient converges to

$$\begin{aligned}\frac{(n-1)!}{2\pi i} \int_C f(\zeta) \left[\frac{1}{(\zeta - z)^2}\right] \left[\frac{n}{(\zeta - z)^{n-1}}\right] d\zeta \\ = \frac{n!}{2\pi i} \int_C \frac{f(\zeta)}{(\zeta - z)^{n+1}} d\zeta,\end{aligned}$$

which completes the induction argument and proves the theorem. □

Cauchy inequality

Corollary

If f is holomorphic in an open set $\Omega \subseteq \mathbb{C}$ that contains a closed disc $\overline{D}(z_0, R)$ centered at z_0 and of radius R, then

$$\left| f^{(n)}(z_0) \right| \leq \frac{n! \|f\|_{L^\infty(C)}}{R^n},$$

where $\|f\|_{L^\infty(C)} = \sup_{z \in C} |f(z)|$ denotes the supremum of $|f|$ on the boundary circle $C = \partial \overline{D}(z_0, R)$.

Proof: Applying the Cauchy integral formula for $f^{(n)}(z_0)$, we obtain

$$\begin{aligned} \left| f^{(n)}(z_0) \right| &= \left| \frac{n!}{2\pi i} \int_C \frac{f(\zeta) d\zeta}{(\zeta - z_0)^{n+1}} \right| \\ &= \frac{n!}{2\pi} \left| \int_0^{2\pi} \frac{f\left(z_0 + Re^{i\theta}\right)}{\left(Re^{i\theta}\right)^{n+1}} Rie^{i\theta} d\theta \right| \leq \frac{n!}{2\pi} \frac{\|f\|_{L^\infty(C)}}{R^n} 2\pi. \end{aligned}$$

□

Cauchy's integral formula and power series

Theorem

Suppose f is holomorphic in an open set $\Omega \subseteq \mathbb{C}$. If D is a disc centered at z_0 and whose closure is contained in Ω, then f has a power series expansion at z_0, given by

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n$$

for all $z \in D$, and the coefficients are given by

$$a_n = \frac{f^{(n)}(z_0)}{n!} \quad \text{for all} \quad n \geq 0.$$

- Observe that since power series define indefinitely (complex) differentiable functions, the theorem gives another proof that a holomorphic function is automatically indefinitely differentiable.

Cauchy's integral formula and power series

Proof: Fix $z \in D$. By the Cauchy integral formula, we have

$$f(z) = \frac{1}{2\pi i} \int_C \frac{f(\zeta)}{\zeta - z} d\zeta,$$

where C denotes the boundary of the disc D and $z \in D$.

- The idea is to write

$$\frac{1}{\zeta - z} = \frac{1}{\zeta - z_0 - (z - z_0)} = \frac{1}{\zeta - z_0} \frac{1}{1 - \left(\frac{z - z_0}{\zeta - z_0}\right)},$$

  and use the geometric series expansion.

- Since $\zeta \in C$ and $z \in D$ is fixed, there exists $0 < r < 1$ such that

$$\left| \frac{z - z_0}{\zeta - z_0} \right| < r.$$

Cauchy's integral formula and power series

- Therefore
$$\frac{1}{1-\left(\frac{z-z_0}{\zeta-z_0}\right)}=\sum_{n=0}^{\infty}\left(\frac{z-z_0}{\zeta-z_0}\right)^n,$$
where the series converges uniformly for $\zeta \in C$.
- This allows us to interchange the infinite sum with the integral when we combine the last three identities, thereby obtaining
$$f(z)=\sum_{n=0}^{\infty}\left(\frac{1}{2\pi i}\int_C \frac{f(\zeta)d\zeta}{(\zeta-z_0)^{n+1}}\right)\cdot(z-z_0)^n.$$
- This proves the power series expansion; further the use of the Cauchy integral formulas for the derivatives (or simply differentiation of the series) proves the formula for a_n. □

Remarks

Remarks

- Another important observation is that the power series expansion of f centered at z_0 converges in any disc, no matter how large, as long as its closure is contained in Ω.
- In particular, if f is entire (that is, holomorphic on all of $\mathbb{C}$), the previous theorem implies that f has a power series expansion around 0, say $f(z) = \sum_{n=0}^{\infty} a_n z^n$, that converges in all of $\mathbb{C}$.

Concluding remarks

Theorem

Let $\Omega \subseteq \mathbb{C}$ be an open set. Then the following statements are exuivalent.

(i) $f \in H(\Omega)$.

(ii) $f^{(n)} \in H(\Omega)$ *for every* $n \geq 0$.

(iii) *If* D *is a disc centered at* z_0 *and whose closure is contained in* Ω, *then* f *has a power series expansion at* z_0, *given by*

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n$$

for all $z \in D$, *and the coefficients are given by*

$$a_n = \frac{f^{(n)}(z_0)}{n!} \quad \text{for all} \quad n \geq 0.$$

Annuli

Definition

Let $0 \leq s_1 < s_2 \leq +\infty$ and $z_0 \in \mathbb{C}$ be given.

- An open annulus centered at z_0 with radii s_1, and s_2 is defined by

$$A(z_0, s_1, s_2) = \{z \in \mathbb{C} : s_1 < |z - z_0| < s_2\}.$$

- A closed annulus centered at z_0 with radii s_1, and s_2 is defined by

$$\overline{A}(z_0, s_1, s_2) = \{z \in \mathbb{C} : s_1 \leq |z - z_0| \leq s_2\}.$$

- Note that

$$\begin{aligned}
A(z_0, s_1, s_2) &= D(z_0, s_2) \setminus \overline{D}(z_0, s_1), \\
\overline{A}(z_0, s_1, s_2) &= \overline{D}(z_0, s_2) \setminus D(z_0, s_1), \\
\overline{A}(z_0, s_1, s_2) &= A(z_0, s_1, s_2) \cup C(z_0, s_1) \cup C(z_0, s_2).
\end{aligned}$$

Laurent series

Definition

Let $z_0 \in \mathbb{C}$. If two series

$$\sum_{n\geq 0} a_n(z-z_0)^n, \quad \text{and} \quad \sum_{n<0} a_n(z-z_0)^n \tag{*}$$

converge for some $z \in \mathbb{C}$, then their sum

$$\sum_{n\in\mathbb{Z}} a_n(z-z_0)^n \tag{**}$$

also converges at $z \in \mathbb{C}$.

- The sum in (**) is called the **Laurent series** centered at z_0.
- The sum (**) is said to **diverge** if at least one of the series in (*) diverges.

Laurent series

Theorem

Let $z_0 \in \mathbb{C}$

$$f(z) = \sum_{n \in \mathbb{Z}} a_n (z - z_0)^n$$

be the Laurent series, and define

$$r = \limsup_{n \to \infty} \sqrt[n]{|a_{-n}|}, \quad \textit{and} \quad R = \left[\limsup_{n \to \infty} \sqrt[n]{|a_n|} \right]^{-1}.$$

If $r < R$*, then the Laurent series* $f(z)$ *converges absolutely on the annulus*

$$A(z_0, r, R) = \{ z \in \mathbb{C} : r < |z - z_0| < R \}$$

and uniformly on its compact subsets. Therefore, the Laurent series $f(z)$ *defines a holomorphic function on* $A(z_0, r, R)$*.*

Laurent series

Proof: We will analyze the series from (*) and (**) separately.

- Note that the first series

$$g(z) = \sum_{n \geq 0} a_n (z - z_0)^n$$

  is the power series centered at $z_0 \in \mathbb{C}$.

- Its radius of convergence is given by the formula

$$R = \left[\limsup_{n \to \infty} \sqrt[n]{|a_n|} \right]^{-1}.$$

- We know that the series converges absolutely on $D\left(z_0, R\right)$, and uniformly on its compact subsets, and diverges on $\overline{D}\left(z_0, R\right)^c$.
- Here we adopt the convention that $1/0 = \infty, 1/\infty = 0$.
- Consequently, $g(z)$ is a holomorphic function on $D\left(z_0, R\right)$.

Laurent series

- If we set $w = \frac{1}{z-z_0}$ then we obtain a new power series

$$h(z) = \sum_{n<0} a_n (z-z_0)^n = \sum_{n\geq 1} a_{-n} w^n,$$

with the corresponding radius of convergence $\rho = r^{-1}$.

- Therefore, the series (of variable w) converges absolutely on $D(0,\rho)$, and uniformly on its compact subsets, and diverges on $\overline{D}(0,\rho)^c$.
- But this is equivalent to say that the original series (of variable z) converges absolutely on $\overline{D}(z_0,r)^c$, and uniformly on its compact subsets, and diverges on $D(z_0,r)$.
- Consequently, $h(z)$ is a holomorphic function on $\overline{D}(z_0,r)^c$.
- Since $r<R$ and $\overline{D}(z_0,r)^c \cap D(z_0,R) = A(z_0,r,R)$ we conclude that $f(z) = g(z) + h(z)$ is holomorphic on $A(z_0,r,R)$ and desired. $\square$

Cauchy integral formula for annuli

Theorem

Let f be analytic on an open set Ω containing the annulus $\overline{A}(z_0, r_1, r_2)$ for $0 < r_1 < r_2 < \infty$, and let γ_1 and γ_2 denote the positively oriented inner and outer boundaries of the annulus. Then for $z \in A(z_0, r_1, r_2)$, we have

$$f(z) = \frac{1}{2\pi i}\int_{\gamma_2} \frac{f(w)}{w-z}dw - \frac{1}{2\pi i}\int_{\gamma_1} \frac{f(w)}{w-z}dw$$

Proof: We can assume that $\overline{A}(z_0, r_1, r_2) \subset A(z_0, s_1, s_2) \subseteq \Omega$ for some $0 < s_1 < r_1 < r_2 < s_2$.

- Note that $A(z_0, s_1, s_2)$ is the region in which the Cauchy integral formula is available. Why?

Cauchy integral formula for annuli

- Let $\Gamma = I_1 \cup C_2 \cup I_2 \cup C_1 \subset A(z_0, s_1, s_2)$ be a path consisting of the following curves:
  - (i) I_1 is the positively oriented interval $[r_1, r_2] \subset \mathbb{R}$ and $I_2 = -I_1 = [r_2, r_1]$;
  - (ii) $C_1 = -\gamma_1$, where γ_1 is the positively oriented circle centered at z_0 and radius r_1; and $C_2 = \gamma_2$ is the positively oriented circle centered at z_0 and radius r_2.
- Then by the Cauchy integral formula for $A(z_0, s_1, s_2)$, we obtain

$$\begin{aligned} f(z) &= \frac{1}{2\pi i}\int_\Gamma \frac{f(w)}{w-z}dw \\ &= \frac{1}{2\pi i}\int_{C_2} \frac{f(w)}{w-z}dw + \frac{1}{2\pi i}\int_{C_1} \frac{f(w)}{w-z}dw \\ &= \frac{1}{2\pi i}\int_{\gamma_2} \frac{f(w)}{w-z}dw - \frac{1}{2\pi i}\int_{\gamma_1} \frac{f(w)}{w-z}dw, \end{aligned}$$

  since

$$\frac{1}{2\pi i}\int_{I_1} \frac{f(w)}{w-z}dw + \frac{1}{2\pi i}\int_{I_2} \frac{f(w)}{w-z}dw = 0. \qquad \square$$

Laurent series representation

Theorem

If f is holomorphic on $\Omega = A(z_0, s_1, s_2)$, then there is a unique two-tailed sequence $(a_n)_{n\in\mathbb{Z}} \subset \mathbb{C}$ such that

$$f(z) = \sum_{n=-\infty}^{\infty} a_n (z - z_0)^n, \qquad \text{for} \quad z \in \Omega.$$

In fact, if r is such that $s_1 < r < s_2$, then the coefficients a_n are given by

$$a_n = \frac{1}{2\pi i} \int_{C(z_0,r)} \frac{f(w)dw}{(w - z_0)^{n+1}}, \qquad \text{for} \quad n \in \mathbb{Z}.$$

Also, the above series converges absolutely on Ω and uniformly on compact subsets of Ω.

Laurent series representation

Proof: It suffices to prove this theorem for any compact subset of Ω. But any compact subset of Ω is contained in closed annulus

$$\overline{A}(z_0, \rho_1, \rho_2)$$

for some ρ_1 and ρ_2 with $s_1 < \rho_1 < \rho_2 < s_2$.

- We choose r_1 and r_2 such that $s_1 < r_1 < \rho_1 < \rho_2 < r_2 < s_2$. Then

$$\overline{A}(z_0, \rho_1, \rho_2) \subset A(z_0, r_1, r_2) \subset \Omega = A\left(z_0, s_1, s_2\right).$$

- By the previous theorem, for any $z \in A(z_0, r_1, r_2)$, we have

$$f(z) = \frac{1}{2\pi i} \int_{C(z_0, r_2)} \frac{f(w)}{w - z} dw - \frac{1}{2\pi i} \int_{C(z_0, r_1)} \frac{f(w)}{w - z} dw.$$

Laurent series representation

- Consider the Cauchy type integral

$$\frac{1}{2\pi i}\int_{C(z_0,r_2)}\frac{f(w)}{w-z}dw \quad \text{for} \quad z\in D(z_0,r_2).$$

- Then proceeding as before, we obtain

$$\frac{1}{2\pi i}\int_{C(z_0,r_2)}\frac{f(w)}{w-z}dw=\sum_{n=0}^{\infty}a_n(z-z_0)^n,$$

where

$$a_n=\frac{1}{2\pi i}\int_{C(z_0,r_2)}\frac{f(w)dw}{(w-z_0)^{n+1}}.$$

- The series converges absolutely on $D(z_0,r_2)$, and uniformly on compact subsets of $D(z_0,r_2)$.

Laurent series representation

- Next, consider the Cauchy type integral

$$-\frac{1}{2\pi i}\int_{C(z_0,r_1)} \frac{f(w)}{w-z}dw \quad \text{for} \quad z \in \overline{D}(z_0,r_1)^c.$$

- This can be written as

$$\begin{aligned}\frac{1}{2\pi i}\int_{C(z_0,r_1)} &\frac{f(w)dw}{(z-z_0)\left[1-\frac{w-z_0}{z-z_0}\right]} \\ &= \frac{1}{2\pi i}\int_{C(z_0,r_1)} \left[\sum_{n=1}^{\infty} f(w)\frac{(w-z_0)^{n-1}}{(z-z_0)^n}\right] dw\end{aligned}$$

- By the Weierstrass M-test, the series converges absolutely and uniformly for $w \in C(z_0,r_1)$.

Laurent series representation

- Consequently, we may integrate term by term to obtain the series

$$-\frac{1}{2\pi i}\int_{C(z_0,r_1)}\frac{f(w)}{w-z}dw=\sum_{n=1}^{\infty}b_n\,(z-z_0)^{-n}\,,$$

where

$$b_n=\frac{1}{2\pi i}\int_{C(z_0,r_1)}\frac{f(w)}{(w-z_0)^{-n+1}}dw.$$

- This is a power series in $1/\left(z-z_0\right)$, that converges for $z\in\overline{D}(z_0,r_1)^c$, and hence uniformly on sets of the form

$$\left\{z\in\mathbb{C}:|z-z_0|\geq 1/\rho\right\},$$

where $(1/\rho)>r_1$. It follows that the convergence is uniform on compact subsets of $\{z\in\mathbb{C}:|z-z_0|>r_1\}$.

Laurent series representation

- The existence part of the theorem now follows from the above computations, and the fact that if $s_1 < r < s_2$ and $k \in \mathbb{Z}$, then

$$\int_{C(z_0,r)} \frac{f(w)dw}{(w-z_0)^{k+1}} = \int_{C(z_0,r_1)} \frac{f(w)dw}{(w-z_0)^{k+1}} = \int_{C(z_0,r_2)} \frac{f(w)dw}{(w-z_0)^{k+1}}.$$

- We turn now to the question of uniqueness. Let $(c_n)_{n\in\mathbb{Z}}$ be a sequence such that

$$f(z) = \sum_{n\in\mathbb{Z}} c_n (z-z_0)^n$$

  for $z \in A(z_0, s_1, s_2)$.

- This series must converge uniformly on compact subsets of $A(z_0, s_1, s_2)$. Why?

Laurent series representation

- Therefore if k is any integer and $s_1 < r < s_2$, then

$$\begin{aligned}
&\frac{1}{2\pi i}\int_{C(z_0,r)} \frac{f(w)dw}{(w-z_0)^{k+1}} \\
&= \frac{1}{2\pi i}\int_{C(z_0,r)} \left[\sum_{n=-\infty}^{\infty} c_n (w-z_0)^{n-k-1}\right] dw \\
&= \sum_{n=-\infty}^{\infty} c_n \frac{1}{2\pi i}\int_{C(z_0,r)} (w-z_0)^{n-k-1}\, dw \\
&= c_k,
\end{aligned}$$

because

$$\frac{1}{2\pi i}\int_{C(z_0,r)} (w-z_0)^{n-k-1}\, dw = \begin{cases} 1 & \text{if } n-k-1=-1, \\ 0 & \text{otherwise.} \end{cases}$$

The theorem is now proved. □