# Lecture 3

## Power series
## Integration over curves

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# Series of complex numbers

- Given a sequence $(w_n)_{n\geq 0} \subseteq \mathbb{C}$, consider the series $\sum_{n=0}^{\infty} w_n$. If

$$\lim_{n\to\infty} \sum_{k=0}^{n} w_k = w$$

  for some $w \in \mathbb{C}$, then we say that the series **converges** to $w$ and write $w = \sum_{n=0}^{\infty} w_n$. Otherwise, the series is said to **diverge**.
- A useful observation is that a series is convergent iff the partial sums $\sum_{k=0}^{n} w_k$ form a Cauchy sequence, that is, $\lim_{m,n\to\infty} \sum_{k=m}^{n} w_k = 0$.
- The series $\sum_{n=0}^{\infty} w_n$ is said to **converge absolutely** if the series $\sum_{n=0}^{\infty} |w_n|$ is convergent.
- As in the real variables case, an absolutely convergent series is convergent.
- A necessary and sufficient condition for absolute convergence is that the sequence of partial sums $\sum_{k=0}^{n} |w_k|$ be bounded.

# Ratio and root tests

**Theorem (Ratio tests)**

*Let $\sum_{n\geq 0} w_n$ be a series of nonzero terms.*

- *If* $\limsup_{n\to\infty} \left|\frac{w_{n+1}}{w_n}\right| < 1$, *then the series converges absolutely.*
- *If* $\left|\frac{w_n+1}{w_n}\right| \geq 1$ *for all sufficiently large* $n$, *the series diverges.*

**Proof:** Exercise! □

**Theorem (Root tests)**

*Let $\sum_{n\geq 0} w_n$ be any complex series.*

- *If* $\limsup_{n\to\infty} |w_n|^{1/n} < 1$, *the series converges absolutely.*
- *If* $\limsup_{n\to\infty} |w_n|^{1/n} > 1$, *the series diverges.*

**Proof:** Exercise! □

# The Weierstrass $M$-Test

**Fact**

Let $(f_n)_{n\in\mathbb{N}}$ be sequence of complex-valued functions on a set $S$.

- Then $(f_n)_{n\in\mathbb{N}}$ **converges pointwise** on $S$ (that is, for each $z \in S$, the sequence $(f_n(z))_{n\in\mathbb{N}}$ is convergent in $\mathbb{C}$ ) iff $(f_n)_{n\in\mathbb{N}}$ is pointwise Cauchy (that is, for each $z \in S$, the sequence $(f_n(z))_{n\in\mathbb{N}}$ is a Cauchy sequence in $\mathbb{C}$).
- Also, $(f_n)_{n\in\mathbb{N}}$ **converges uniformly** iff $(f_n)_{n\in\mathbb{N}}$ is uniformly Cauchy on $S$, in other words, $\lim_{m,n\to\infty} |f_n(z) - f_m(z)| = 0$, uniformly for $z \in S$.

**Theorem (The Weierstrass $M$-Test)**

*Let $(g_n)_{n\in\mathbb{N}}$ be a sequence complex-valued functions on a set $S \subseteq \mathbb{C}$, and assume that $|g_n(z)| \leq M_n$ for all $z \in S$. If $\sum_{n=1}^{\infty} M_n < +\infty$, then the series $\sum_{n=1}^{\infty} g_n(z)$ converges uniformly on $S$.*

**Proof:** Let $f_n = \sum_{k=1}^{n} g_k$. Then $|f_n - f_m| \leq \sum_{k=m+1}^{n} |g_k| \leq \sum_{k=m+1}^{n} M_k$ for $n > m$. Thus $(f_n)_{n\in\mathbb{N}}$ is uniformly Cauchy on $S$ and we are done. □

# Power series

- The series of the form $\sum_{n=0}^{\infty} a_n (z - z_0)^n$, where $z_0, a_n \in \mathbb{C}$ are called the **power series**. Thus we are dealing with series of functions $\sum_{n=0}^{\infty} f_n$ of a very special type, namely $f_n(z) = a_n (z - z_0)^n$.

**Theorem**

*If $\sum_{n=0}^{\infty} a_n (z - z_0)^n$ converges at the point $z \in \mathbb{C}$ with $|z - z_0| = r$, then the series converges absolutely on $D(z_0, r)$, uniformly on each closed subdisk of $D(z_0, r)$, hence uniformly on each compact subset of $D(z_0, r)$.*

**Proof:** We have $|a_n (z' - z_0)^n| = |a_n (z - z_0)^n| \left|\frac{z'-z_0}{z-z_0}\right|^n$.

- The convergence at $z$ implies that the sequence $(a_n (z - z_0)^n)_{n\in\mathbb{N}}$ is bounded, since $\lim_{n\to\infty} a_n (z - z_0)^n = 0$.
- If $|z' - z_0| \leq r' < r$, then $\left|\frac{z'-z_0}{z-z_0}\right| \leq \frac{r'}{r} < 1$ proving absolute convergence at $z'$ (by comparison with a geometric series). Thus the series converges uniformly on $\overline{D}(z_0, r')$ by the Weierstrass $M$-test. □

# Radius of convergence

**Theorem**

*Let $\sum_{n=0}^{\infty} a_n (z - z_0)^n$ be a power series. Let $r = \left[\limsup_{n\to\infty} \sqrt[n]{|a_n|}\right]^{-1}$, be the* **radius of convergence** *of the series. (Adopt the convention that $1/0 = \infty, 1/\infty = 0$.) The series converges absolutely on $D(z_0, r)$, and uniformly on its compact subsets. The series diverges for $|z - z_0| > r$.*

**Proof:** We have $\limsup_{n\to\infty} |a_n (z - z_0)^n|^{1/n} = |z - z_0| / r$, which will be less than 1 if $|z - z_0| < r$.

- By the root test, the series converges absolutely on $D(z_0, r)$. Uniform convergence on compact subsets follows from the previous result. (We do not necessarily have convergence for $|z - z_0| = r$, but we do have convergence for $|z - z_0| = r'$, for any $r' \in (0, r)$.
- If the series converges at some point $z$ with $|z - z_0| > r$, then by the previous theorem it converges absolutely at points $z'$ so that $r < |z' - z_0| < |z - z_0|$. But then $|z - z_0| / r > 1$, contradicting the root test. □

# Power series are holomorphic

## Definition

Let $\Omega \subseteq \mathbb{C}$ be an open set. We say that a function $f : \Omega \to \mathbb{C}$ is **representable by power series** in $\Omega$ if to every disc $D(a, r) \subseteq \Omega$ we have

$$f(z) = \sum_{n=0}^{\infty} c_n (z - a)^n \quad \text{for all} \quad z \in D(a, r). \tag{*}$$

## Theorem

*Let $\Omega \subseteq \mathbb{C}$ be an open set. If $f : \Omega \to \mathbb{C}$ is representable by power series in $\Omega$, then $f \in H(\Omega)$ and $f'$ is also representable by power series in $\Omega$. In fact, if (*) holds, then we also have*

$$f'(z) = \sum_{n=1}^{\infty} n c_n (z - a)^{n-1} \quad \textit{for all} \quad z \in D(a, r). \tag{**}$$

# Power series are holomorphic

**Proof:** The key idea of the proof is to compare the differential quotient

$$\frac{f(z) - f(w)}{z - w}$$

with the power series (**) evaluated at $w$. Then, we let $z \to w$.

- If the series (*) converges in $D(a, r)$, then by the root test, the series (**) also converges in that domain.
- Without loss of generality, set $a = 0$ and let

$$g(z) = \sum_{n=1}^{\infty} n c_n z^{n-1}.$$

- Fix $w \in D(0, r)$ and choose $\rho$ such that $|w| < \rho < r$. We have

$$\frac{f(z) - f(w)}{z - w} - g(w) = \sum_{n=1}^{\infty} c_n \left[ \frac{z^n - w^n}{z - w} - n w^{n-1} \right] \quad \text{if} \quad z \neq w.$$

# Power series are holomorphic

- For $n = 1$, the expression in brackets equals 0. For $n \geq 2$, it becomes:

$$\left[\frac{z^n - w^n}{z - w} - nw^{n-1}\right] = (z - w)\sum_{k=1}^{n-1} kw^{k-1}z^{n-k-1}. \tag{1}$$

- This follows from the formula

$$z^n - w^n = (z - w)\sum_{k=0}^{n-1} z^{n-1-k}w^k,$$

and the telescoping identity

$$\begin{aligned}\sum_{k=0}^{n-1} z^{n-1-k}w^k &= \sum_{k=0}^{n-1}((k+1) - k)z^{n-1-k}w^k \\ &= \sum_{k=1}^{n} kz^{n-k}w^{k-1} - \sum_{k=1}^{n-1} kz^{n-1-k}w^k.\end{aligned}$$

- If $|z| < \rho$, then

$$\left| \sum_{k=1}^{n-1} k w^{k-1} z^{n-k-1} \right| \leq \frac{n(n-1)}{2} \rho^{n-2}.$$

- Thus, we have

$$\left| \frac{f(z) - f(w)}{z - w} - g(w) \right| \leq |z - w| \sum_{n=2}^{\infty} n^2 |c_n| \rho^{n-2}.$$

- Since $\rho < r$, the last series converges.
- Hence,

$$\lim_{z \to w} \left| \frac{f(z) - f(w)}{z - w} - g(w) \right| = 0.$$

- This implies that $f'(w) = g(w)$, completing the proof. □

# Power series are holomorphic

## Remark

Since $f'$ satisfies the same conditions as $f$, the theorem can be applied to $f'$ as well. This implies that $f$ has derivatives of all orders, each of which can be represented by a power series in $\Omega$.

- Specifically, if (*) holds, then

$$f^{(k)}(z) = \sum_{n=k}^{\infty} n(n-1)\cdots(n-k+1)c_n(z-a)^{n-k}.$$

- Consequently, (*) leads to

$$k!c_k = f^{(k)}(a) \quad \text{for} \quad k = 0, 1, 2, \ldots,$$

ensuring that for each $a \in \Omega$, there exists a unique sequence $(c_n)_{n\geq 0}$ satisfying (*).

# Examples

- The geometric series
$$\sum_{n=0}^{\infty} z^n$$
converges absolutely only within the disk $|z| < 1$.
- Its sum within this region is the function $\frac{1}{1-z}$, which is holomorphic in the open set $\mathbb{C} \setminus \{1\}$.
- This identity is established similarly to the real case:
$$\sum_{n=0}^{N} z^n = \frac{1 - z^{N+1}}{1 - z}.$$
Then $\lim_{N\to\infty} z^{N+1} = 0$ if $|z| < 1$.
- By the previous theorem, for $z \in D(0,1)$, we have
$$\frac{1}{(1-z)^2} = \left(\frac{1}{1-z}\right)' = \sum_{n=1}^{\infty} n z^{n-1}.$$

# Examples

- The most important example of a power series is the complex exponential function, which is defined for $z \in \mathbb{C}$ by
$$\exp(z) = \sum_{n=0}^{\infty} \frac{z^n}{n!}.$$
When $z$ is real, this definition coincides with the usual exponential function and in fact, the series above converges absolutely for every $z \in \mathbb{C}$.
- Further examples of power series that converge in the whole complex plane are given by the standard trigonometric functions; these are defined by
$$\cos z = \sum_{n=0}^{\infty} (-1)^n \frac{z^{2n}}{(2n)!}, \quad \text{and} \quad \sin z = \sum_{n=0}^{\infty} (-1)^n \frac{z^{2n+1}}{(2n+1)!},$$
and they agree with the usual cosine and sine of a real argument whenever $z \in \mathbb{R}$.

# More about $\exp(z), \sin(z)$ and $\cos(z)$

- In order to show that the series defining $\exp(z)$ converges absolutely, observe that
$$\left|\frac{z^n}{n!}\right| = \frac{|z|^n}{n!}.$$
Thus, $|\exp(z)|$ can be compared to the series $\sum_{n=0}^{\infty} \frac{|z|^n}{n!} = e^{|z|} < \infty$. In fact, this estimate shows that the series defining $\exp(z)$ is uniformly convergent in every disk in $\mathbb{C}$.
- A similar argument can be used to deduce the convergence of power series for $\sin z$ and $\cos z$.
- By the previous theorem, for any $z \in \mathbb{C}$, the complex derivative of $\exp(z)$ exists and is given by
$$\exp'(z) = \sum_{n=0}^{\infty} n\frac{z^{n-1}}{n!} = \sum_{m=0}^{\infty} \frac{z^m}{m!} = \exp(z),$$
therefore $\exp(z)$ is its own derivative.

# More about $\exp(z), \sin(z)$ and $\cos(z)$

- A similar argument gives us that $\cos' z = -\sin z$ and $\sin' z = \cos z$. This shows that these are entire functions as well.
- Since the series defining the exponential function is absolutely convergent, we may multiply it with itself to obtain that for $z, w \in \mathbb{C}$, we have

$$\left(\sum_{k=0}^{\infty} \frac{z^k}{k!}\right)\left(\sum_{m=0}^{\infty} \frac{w^m}{m!}\right) = \sum_{n=0}^{\infty} \frac{1}{n!} \sum_{k=0}^{n} \frac{n!}{k!(n-k)!} z^k w^{n-k}$$
$$= \sum_{n=0}^{\infty} \frac{(z+w)^n}{n!},$$

  which shows that

$$\exp(z)\exp(w) = \exp(z+w). \tag{A}$$

- A simple calculation shows that for $y \in \mathbb{R}$ we have

$$\exp(iy) = \cos(y) + i\sin(y). \tag{B}$$

# More about $\exp(z), \sin(z)$ and $\cos(z)$

- We have $\exp(z) = 1$ if and only if $z = 2k\pi i$. Indeed, let $z = x + iy$ and if $\exp(z) = 1$, then $|\exp(z)| = 1$.
- The identities (A) and (B), together with the Pythagorean trigonometric identity, imply that $x = 0$. Hence, $z = iy$.
- If $\exp(iy) = 1$, this implies that $\cos(y) = 1$ and $\sin(y) = 0$. This implies that $y = 2k\pi$, where $k \in \mathbb{Z}$. Therefore, $z = 2k\pi i$.
- Consequently, by (A), we get that the complex exponential is a periodic function with period $2\pi i$. This implies that

$$\exp(z + 2k\pi i) = \exp(z), \quad \text{for all} \quad z \in \mathbb{C} \text{ and } k \in \mathbb{Z}.$$

# Continuous curves in topological space

**Definition**

If $X$ is a topological space, a **curve** in $X$ is a continuous mapping $\gamma$ of a compact interval $[\alpha, \beta] \subset \mathbb{R}$ into $X$; here $\alpha < \beta$. We call $[\alpha, \beta]$ the parameter interval of $\gamma$ and denote the range of $\gamma$ by

$$\gamma^* = \gamma([\alpha, \beta]) = \{\gamma(t) : t \in [\alpha, \beta]\}.$$

- Observe that $\gamma^*$ is compact and connected.
- If the initial point $\gamma(\alpha)$ of $\gamma$ coincides with its end point $\gamma(\beta)$, we call $\gamma$ a **closed curve**.

- In the definition of a curve in $\mathbb{C}$, we will distinguish between the one-dimensional geometric object in the plane (endowed with an orientation) $\gamma^*$, and its parametrization $\gamma$, which is a mapping from a closed interval to $\mathbb{C}$. This parametrization is not uniquely determined.

# Equivalent curves

- Two parametrizations,

$$\gamma : [\alpha, \beta] \to \mathbb{C} \quad \text{and} \quad \tilde{\gamma} : [\alpha_1, \beta_1] \to \mathbb{C},$$

  are equivalent if there exists a continuously differentiable bijection from $[\alpha_1, \beta_1] \ni s \mapsto \varphi(s) \in [\alpha, \beta]$ such that $\varphi'(s) > 0$ and

$$\tilde{\gamma}(s) = \gamma(\varphi(s)).$$

- The condition $\varphi'(s) > 0$ precisely ensures that the orientation is preserved: as $s$ travels from $\alpha_1$ to $\beta_1$, $\varphi(s)$ travels from $\alpha$ to $\beta$.
- The family of all parametrizations that are equivalent to $\gamma(t)$ determines a smooth curve $\gamma^* \subset \mathbb{C}$, namely the image of $[\alpha, \beta]$ under $\gamma$, with the orientation given by $\gamma$ as $t$ travels from $\alpha$ to $\beta$.

# Path in topological space

### Definition

A **path** is a piecewise continuously differentiable curve in the plane. More precisely, a path with parameter interval $[\alpha, \beta]$ is a continuous complex function $\gamma$ defined on $[\alpha, \beta]$, satisfying the following conditions:

- There exist finitely many points $s_j$ such that

$$\alpha = s_0 < s_1 < \cdots < s_n = \beta,$$

  and on each interval $[s_{j-1}, s_j]$, the function $\gamma$ has a continuous derivative.
- However, at the points $s_1, \ldots, s_{n-1}$, the left-hand and right-hand derivatives of $\gamma$ may differ.
- A **closed path** is a closed curve that is also a path.

# Integration over paths

**Definition**

Let $\gamma$ be a path with parameter interval $[\alpha, \beta]$. Assume that $f$ is continuous on $\gamma^* \subset \mathbb{C}$. Then we define

$$\int_\gamma f(z)dz = \int_\alpha^\beta f(\gamma(t))\gamma'(t)dt.$$

- When $\gamma$ is closed path, then integration over $\gamma$ is understood to be in the anticlockwise direction, unless otherwise mentioned.
- Here, the integral on the right-hand side is the Riemann integral since $\gamma'(t)$ is a bounded function of $t$ in $[\alpha, \beta]$ with at most finitely many discontinuities.

# Properties of the integrals over paths

(i) Let $\phi : [a, b] \to [\alpha, \beta]$ be continuous, strictly increasing and onto function. Further assume that $\phi$ is continuously differentiable.

- Then $\phi(a) = \alpha, \phi(b) = \beta$ and $\phi([a, b]) = [\alpha, \beta]$.
- Let $\gamma$ be a path with parameter interval $[\alpha, \beta]$. Then

$$\sigma = \gamma \circ \phi$$

  is a path with parameter interval $[a, b]$.
- Let $f$ be continuous on $\gamma^*$. Then $f$ is also continuous on $\sigma^*$ and

$$\int_\sigma f(z)dz = \int_\gamma f(z)dz.$$

- We call $\phi : [a, b] \to [\alpha, \beta]$ a change of parameter function.

# Properties of the integrals over paths

(ii) Let $\gamma_1$ and $\gamma_2$ be paths such that the end point of $\gamma_1$ coincides with the initial point of $\gamma_2$. Then, after suitable re-parametrization, we obtain a path $\gamma$ by first following $\gamma_1$ and then $\gamma_2$.

- By (i), we have

$$\int_\gamma f(z)dz = \int_{\gamma_1} f(z)dz + \int_{\gamma_2} f(z)dz,$$

where $f$ is continuous on $\gamma_1^* \cup \gamma_2^*$. We write $\gamma = \gamma_1 + \gamma_2$.

- For paths $\gamma_1, \gamma_2, \ldots, \gamma_n$ such that the end points of $\gamma_j$ coincides with the initial point of $\gamma_{j+1}$ with $1 \leq j < n$, the path

$$\gamma = \gamma_1 + \gamma_2 + \cdots + \gamma_n$$

is defined similarly.

# Properties of the integrals over paths

(iii) Let $\gamma : [0,1] \to \mathbb{C}$ be a path. Define $\gamma_1(t) = \gamma(1-t)$ for $t \in [0,1]$. Then $\gamma_1$ is called a path opposite to $\gamma$. We have

$$\int_\gamma f(z)dz = -\int_{\gamma_1} f(z)dz,$$

where $f$ is continuous on $\gamma^*$.

(iv) Let $\gamma : [\alpha, \beta] \to \mathbb{C}$ be a path and $f$ be continuous on $\gamma^*$. Then

$$\begin{aligned}\left|\int_\gamma f(z)dz\right| &= \left|\int_\alpha^\beta f(\gamma(t))\gamma'(t)dt\right| \\ &\le \max_{z\in\gamma^*}|f(z)| \int_\alpha^\beta |\gamma'(t)|\, dt \\ &= \ell(\gamma) \max_{z\in\gamma^*}|f(z)|\end{aligned}$$

where $\ell(\gamma) = \int_\alpha^\beta |z'(t)|\, dt$ is the **length** of $\gamma$.

# Some remarks

Let $\gamma$ be a closed path.

- Then the complement of $\gamma^*$ in metric space $\mathbb{C}_\infty$ is open. Thus it is a disjoint union of regions, since every open set is a disjoint union of open and connected sets.
- We say that these regions are determined by $\gamma$ in $\mathbb{C}_\infty$. There is only one region determined by $\gamma$ which is unbounded and we call it the unbounded region determined by $\gamma$. We observe that it contains $\infty$.
- The regions determined by $\gamma$ in $\mathbb{C}_\infty$ and the regions determined by $\gamma$ in $\mathbb{C}$ are identical except that the unbounded region determined by $\gamma$ in $\mathbb{C}$ does not contain $\infty$.

# Examples

- If $a$ is a complex number and $r > 0$, the path defined by

$$\gamma(t) := a + re^{it}, \quad 0 \le t \le 2\pi,$$

is called the **positively oriented circle** with center at $a$ and radius $r$ and then we have

$$\int_\gamma f(z)\,\mathrm{d}z = \int_0^{2\pi} f\left(a + re^{it}\right) ire^{it} dt$$

and the length of $\gamma$ is $2\pi r$, as expected.

# Examples

- If $a$ and $b$ are complex numbers, the path $\gamma$ given by

$$\gamma(t) := a + (b-a)t, \quad 0 \le t \le 1,$$

is the **positively oriented interval** $[a,b]$; its length is $|b-a|$, and

$$\int_{[a,b]} f(z)\,\mathrm{d}z = (b-a)\int_0^1 f\big(a+(b-a)t\big)\,dt.$$

Let $\alpha < \beta$ be real numbers. If

$$\gamma_1(t) := \frac{a(\beta-t)+b(t-\alpha)}{\beta-\alpha}, \quad \alpha \le t \le \beta,$$

then we obtain an equivalent path, which we still denote by $[a,b]$.

- The path opposite to $[a,b]$ is $[b,a]$.

# Examples

- Let $\{a, b, c\}$ be an ordered triple of complex numbers, and let

$$\Delta = \Delta(a, b, c)$$

be the triangle with vertices at $a$, $b$, and $c$. The set $\Delta$ is the smallest convex set that contains $a$, $b$, and $c$. Define

$$\int_{\partial\Delta} f = \int_{[a,b]} f + \int_{[b,c]} f + \int_{[c,a]} f \qquad (\triangle)$$

for any $f$ continuous on the boundary of $\Delta$. We can regard $(\triangle)$ as the definition of its left side. Alternatively, we can consider $\partial\Delta$ as a path obtained by joining $[a, b]$ to $[b, c]$ to $[c, a]$, as outlined in definition of the path, in which case $(\triangle)$ is easily proved to be true.

- If $\{a, b, c\}$ is permuted cyclically, we see from $(\triangle)$ that the left side of $(\triangle)$ is unaffected. If $\{a, b, c\}$ is replaced by $\{a, c, b\}$, then the left side of $(\triangle)$ changes sign.