Lecture 21

The prime number theorem

MATH 503, FALL 2025

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Stirling formula

Corollary

Let $0 < \delta < \pi$, then for any $z \in \mathbb{C}$ so that $|\arg z| < \pi - \delta$, we have

$$\log \Gamma(z) = \left(z - \frac{1}{2}\right) \log z - z + \frac{1}{2} \log 2\pi + O(|z|^{-1}),$$

and also

$$\frac{\Gamma'(z)}{\Gamma(z)} = \log z - \frac{1}{2z} + O(|z|^{-2}),$$

uniformly as $|z| \to \infty$, where logarithm has principal value, and the implicit constant depend at most on δ.

Expansion of logarithmic derivative of ζ

Lemma

Let $s = \sigma + it$ with $-1 \leqslant \sigma \leqslant 2$ and t not equal to an ordinate of a zero of $\zeta(s)$. Then we have

$$-\frac{\zeta'(s)}{\zeta(s)} = \frac{1}{s-1} - \sum_{\rho : |t - \operatorname{Im}\rho| \leqslant 1} \frac{1}{s-\rho} + O(\log(|t|+3)),$$

where the summation runs through all the non-trivial zeros of $\zeta(s)$.

Proof: Recall that $\xi(s) = s(s-1)\pi^{-s/2}\Gamma(s/2)\zeta(s)$. Set $\tau = |t| + 3$.

- The logarithmic differentiation yields

  $$\frac{\xi'(s)}{\xi(s)} = b + \sum_{\rho} \left(\frac{1}{s-\rho} + \frac{1}{\rho} \right), \quad \text{where} \quad b = \log(2\sqrt{\pi}) - 1 - \frac{\gamma}{2},$$

  where the summation runs through all zeros $\rho = \beta + i\gamma$ of $\xi(s)$, which are exactly the non-trivial zeros of $\zeta(s)$.

Expansion of logarithmic derivative of ζ

- Moreover, the above sum is absolutely convergent, since

$$\sum_{\rho} |\rho|^{-2} < \infty,$$

as ξ is an entire function of order 1.

- Now using the definition of $\xi(s) = s(s-1)\pi^{-s/2}\Gamma(s/2)\zeta(s)$, we have

$$\frac{\xi'(s)}{\xi(s)} = \frac{1}{s} + \frac{1}{s-1} - \frac{\log \pi}{2} + \frac{\zeta'(s)}{\zeta(s)} + \frac{1}{2}\frac{\Gamma'(s/2)}{\Gamma(s/2)}.$$

- By the logarithmic differentiation of $\Gamma(s)$, we obtain

$$-\frac{\Gamma'(s)}{\Gamma(s)} = \frac{1}{s} + \gamma + \sum_{n=1}^{\infty}\left(\frac{1}{n+s} - \frac{1}{n}\right),$$

since

$$\Gamma(s) = s^{-1}e^{-\gamma s}\prod_{n=1}^{\infty}\left(1 + \frac{s}{n}\right)^{-1} e^{s/n}.$$

Expansion of logarithmic derivative of ζ

- Therefore, we may write

$$\frac{\zeta'(s)}{\zeta(s)} = -\frac{1}{s-1} + \frac{\log \pi}{2} + \frac{\gamma}{2} + \sum_{n=1}^{\infty} \left(\frac{1}{2n+s} - \frac{1}{2n} \right) + b + \sum_{\rho} \left(\frac{1}{s-\rho} + \frac{1}{\rho} \right).$$

- Note that

$$\sum_{1 \le n \le \tau} \left| \frac{1}{2n+s} - \frac{1}{2n} \right| = O(\log \tau), \quad \text{and} \quad \sum_{n \ge \tau} \left| \frac{1}{2n+s} - \frac{1}{2n} \right| = O\left(\frac{|s|}{\tau} \right).$$

- Therefore, we can write that

$$\frac{\zeta'(s)}{\zeta(s)} = -\frac{1}{s-1} + \sum_{\rho} \left(\frac{1}{s-\rho} + \frac{1}{\rho} \right) + O(\log \tau). \qquad (*)$$

- Notice that

$$\left| \frac{\zeta'(2+it)}{\zeta(2+it)} \right| \le \sum_{n=1}^{\infty} \frac{\Lambda(n)}{n^2} = O(1).$$

Expansion of logarithmic derivative of ζ

- Applying (*) with $s = 2 + it$ and using the previous bound, we obtain

$$\left| \sum_\rho \left(\frac{1}{2 + it - \rho} + \frac{1}{\rho} \right) \right| = O(\log \tau).$$

- Adding and subtracting the sum $\sum_\rho \left(\frac{1}{2+it-\rho} + \frac{1}{\rho} \right)$ from (*), we obtain

$$\frac{\zeta'(s)}{\zeta(s)} = -\frac{1}{s-1} + \sum_\rho \left(\frac{1}{s-\rho} - \frac{1}{2+it-\rho} \right) + O(\log \tau).$$

- Note that the zero $\rho = \beta + i\gamma$, satisfies

$$\mathrm{Re}\left(\frac{1}{2+it-\rho} \right) = \frac{2-\beta}{(2-\beta)^2 + (t-\gamma)^2} \geq \frac{1}{4 + 4(t-\gamma)^2} \geq 0,$$
$$\mathrm{Re}\left(\frac{1}{\rho} \right) = \frac{\beta}{\beta^2 + \gamma^2} \geq 0.$$

Expansion of logarithmic derivative of ζ

- Therefore, we obtain

$$\sum_{\rho} \frac{1}{4+4(t-\gamma)^2} \leq \mathrm{Re}\left(\sum_{\rho} \left(\frac{1}{2+it-\rho} + \frac{1}{\rho} \right) \right) = O(\log \tau).$$

- This immediately implies that

$$\sum_{|\operatorname{Im}\rho - t| \leq 1} 1 \leq \sum_{|\operatorname{Im}\rho - t| \leq 1} \frac{2}{1+(t-\operatorname{Im}\rho)^2} = O(\log \tau).$$

- In other words, the number of zeros ρ in the strip $t \leq \operatorname{Im}\rho \leq t+1$ is at most $O(\log \tau)$ for any $t \geq 2$.
- By the previous observation we see that

$$\sum_{\rho:|t-\gamma|\leqslant 1} \frac{1}{|2+it-\rho|} = O\Big(\sum_{\rho:|t-\gamma|\leqslant 1} 1 \Big) = O(\log \tau). \qquad (**)$$

Expansion of logarithmic derivative of ζ

- By the previous observation we also have

$$\sum_{\rho:|t-\gamma|>1} \left|\frac{1}{s-\rho} - \frac{1}{2+it-\rho}\right| = O(\log \tau) \qquad (***)$$

- In order to see (***), we split $\sum_{\rho:|t-\gamma|>1} = \sum_{k\in\mathbb{Z}_+} \sum_{\rho:k<|t-\gamma|\leq k+1}$, and observe, arguing as in (**), that for each $k \in \mathbb{Z}_+$ the number of zeros ρ obeying $k < |t-\gamma| \leq k+1$ is at most $O(\log(\tau+k))$. Now let $k \in \mathbb{Z}_+$ and consider the zeros ρ satisfying $k < |\gamma - t| \leqslant k+1$. Since

$$\left|\frac{1}{s-\rho} - \frac{1}{2+it-\rho}\right| = \frac{2-\sigma}{|(s-\rho)(2+it-\rho)|} \leqslant \frac{3}{|\gamma-t|^2} \leqslant \frac{3}{k^2}$$

we infer that the contribution from the sum $\sum_{\rho:k<|t-\gamma|\leq k+1}$ is at most $O(k^{-2}\log(\tau+k))$. Summing over $k \in \mathbb{Z}_+$ we obtain (***).

Expansion of logarithmic derivative of ζ

- Finally, combining (**) and (***) with

$$\frac{\zeta'(s)}{\zeta(s)} = -\frac{1}{s-1} + \sum_{\rho}\left(\frac{1}{s-\rho} - \frac{1}{2+it-\rho}\right) + O(\log \tau),$$

we obtain

$$-\frac{\zeta'(s)}{\zeta(s)} = \frac{1}{s-1} - \sum_{\substack{\rho \\ |t - \operatorname{Im}\rho| \leqslant 1}} \frac{1}{s-\rho} + O(\log \tau),$$

as desired. □

From the proof of the previous lemma, we obtain the following result.

Corollary

For every real number $T \geq 2$ the number of nontrivial zeros ρ of the zeta function ζ satisfying $T \leq \operatorname{Im}\rho \leq T+1$ is at most $O(\log T)$.

Expansion of logarithmic derivative of ζ

Corollary

Let $s = \sigma + it$ and assume that $|s + 2m| \geq 1/2$ for every $m \in \mathbb{N}$. Then

$$-\frac{\zeta'(s)}{\zeta(s)} = \frac{1}{s-1} - \sum_{\rho : |t - \operatorname{Im}\rho| \leqslant 1} \frac{1}{s-\rho} + O(\log(|t|+3)),$$

where the summation runs through all the non-trivial zeros of $\zeta(s)$.

Proof: By the functional equation

$$\zeta(s) = 2^s \pi^{s-1} \sin\left(\frac{\pi s}{2}\right) \Gamma(1-s)\zeta(1-s),$$

and the logarithmic differentiation, we have

$$\frac{\zeta'(s)}{\zeta(s)} = \log(2\pi) + \frac{\pi}{2}\cot\left(\frac{\pi s}{2}\right) - \frac{\Gamma'(1-s)}{\Gamma(1-s)} - \frac{\zeta'(1-s)}{\zeta(1-s)}.$$

Expansion of logarithmic derivative of ζ

- Except for the logarithmic derivative of the gamma function, the terms on the right-hand side are uniformly bounded in the half plane $\operatorname{Re} s \leq -1$ after removing neighborhoods is $|s - 2m| < 1/2$ of the even integers $2m$, these being poles of $\cot(\pi s/2)$.
- Then by Stirling's formula for $\operatorname{Re} s \leq -1$, we obtain

$$\left|\frac{\zeta'(s)}{\zeta(s)}\right| = O(\log(1 + |s|)), \quad \text{since} \quad \left|\frac{\Gamma'(1-s)}{\Gamma(1-s)}\right| = O(\log(1 + |s|)).$$

- For $\operatorname{Re} s \geq 2$, we have

$$\left|\frac{\zeta'(s)}{\zeta(s)}\right| = O(1).$$

- Also if $\operatorname{Re} s \leq -1$ or $\operatorname{Re} s \geq 2$, then we have

$$\frac{1}{s-1} - \sum_{\rho : |t - \operatorname{Im} \rho| \leqslant 1} \frac{1}{s - \rho} = O(\log(|t| + 3)).$$

- If we combine this with the previous lemma the proof follows. □

Some quantitative bounds

Corollary

For every real number $T \geqslant 2$, there exists $T' \in [T, T+1]$ such that, uniformly for $-1 \leqslant \sigma \leqslant 2$, we have

$$\left|\frac{\zeta'(\sigma + iT')}{\zeta(\sigma + iT')}\right| = O(\log^2 T).$$

Proof: We subdivide $[T, T+1]$ into $O(\log T)$ equal parts of length $c/\log T$, where $c > 0$ is chosen so that the number of parts exceeds the number of zeros.

- By the Dirichlet pigeonhole principle, we deduce that there is a part that contains no zeros. Hence for T' lying in this part, we must have $|T' - \gamma| \geq c'/\log T$ for some $c' > 0$.
- We infer that each summand in the previous lemma is $O(\log T)$ and since there are $O(\log T)$ summands by the previous corollary, we obtain the desired estimate. This completes the proof. □

Zero-free region estimates

Theorem (de la Vallée Poussin)

There exists an absolute constant $C > 0$ such that $\zeta(s)$ has no zero $\rho = \beta + i\gamma$ satisfying

$$\beta \geq 1 - \frac{C}{\log(|\gamma| + 2)}. \tag{*}$$

Proof: At the point $s = 1$ the zeta function $\zeta(s)$ has a pole, and so there exists $c_1 \in \mathbb{R}_+$ so that $\zeta(s)$ has no zeros in the domain $|s - 1| < 2c_1$. Thus if $\rho = \beta + i\gamma$ is a nontrivial zero of $\xi(s)$ then $|\rho - 1| \geq 2c_1$. If $|\gamma| \leq c_1$, then $1 - \beta \geq c_1 \geq \frac{c_1}{4\log 2} \geq \frac{c_1}{4\log(|\gamma|+2)}$ implying (*) with $C = c_1/4$.

- We now fix a particular zero $\rho_0 = \beta_0 + i\gamma_0$ of $\zeta(s)$ such that $|\gamma_0| > c_1$.
- Suppose that $s = \sigma + it$ with $\sigma > 1$. Taking real parts we obtain

$$-\operatorname{Re}\left(\frac{\zeta'(s)}{\zeta(s)}\right) = \sum_{n=1}^{\infty} \Lambda(n) n^{-\sigma} \cos(t \log n).$$

Zero-free region estimates

- Since $3 + 4\cos\theta + \cos 2\theta = 2(1 + \cos\theta)^2 \geq 0$ for any $\theta \in \mathbb{R}$, we have
$$-3\frac{\zeta'(\sigma)}{\zeta(\sigma)} - 4\,\mathrm{Re}\left(\frac{\zeta'(\sigma + it)}{\zeta(\sigma + it)}\right) - \mathrm{Re}\left(\frac{\zeta'(\sigma + 2it)}{\zeta(\sigma + 2it)}\right) \geq 0. \qquad (**)$$
- Since $\zeta(s)$ has a pole of residue 1 at $s = 1$, we have
$$-\frac{\zeta'(\sigma)}{\zeta(\sigma)} = \frac{1}{\sigma - 1} + O(1).$$
- We consider $s = \sigma + it$ with $t = \gamma_0$. Since $|\gamma_0| \geq c_1 > 0$, we have
$$\begin{aligned} -\mathrm{Re}\left(\frac{\zeta'(\sigma + i\gamma_0)}{\zeta(\sigma + i\gamma_0)}\right) &\leq -\mathrm{Re}\sum_{|\gamma - \gamma_0| \leq 1} \frac{1}{(\sigma - \beta) + i(\gamma_0 - \gamma)} \\ &\quad + c_2 \log(|\gamma_0| + 2) \\ &\leq \frac{-1}{(\sigma - \beta_0)} + c_2 \log(|\gamma_0| + 2), \end{aligned}$$

  by proceeding as in the previous lemma.

Zero-free region estimates

- Similarly, we have

$$-\operatorname{Re}\left(\frac{\zeta'\left(\sigma+2i\gamma_0\right)}{\zeta\left(\sigma+2i\gamma_0\right)}\right)\leq c_3\log\left(|\gamma_0|+2\right).$$

- Inserting these three estimates into (**), we deduce that for σ close to 1,

$$4\left(\sigma-\beta_0\right)^{-1}-3(\sigma-1)^{-1}\leq c_4\log\left(|\gamma_0|+2\right)$$

- Choosing $\sigma=1+\frac{1}{2c_4\log(|\gamma_0|+2)}$, we obtain

$$\beta_0\leq 1-\frac{1}{14c_4\log\left(|\gamma_0|+2\right)},$$

which establishes (*) when $|\gamma_0|\geq c_1$. □

Important estimates

Lemma

Let $\kappa, T, T' \in \mathbb{R}_+$ *be given, and consider the following function*

$$h(x) = \begin{cases} 1 & \text{if } x \in (1,\infty), \\ \frac{1}{2} & \text{if } x = 1, \\ 0 & \text{if } x \in (0,1). \end{cases}$$

- *If* $x \neq 1$, *then*

$$\left| h(x) - \frac{1}{2\pi i} \int_{\kappa - iT'}^{\kappa + iT} x^s \frac{\mathrm{d}s}{s} \right| \leq \frac{x^\kappa}{2\pi |\log x|} \left(\frac{1}{T} + \frac{1}{T'} \right).$$

- *If* $x = 1$, *then*

$$\left| h(1) - \frac{1}{2\pi i} \int_{\kappa - iT}^{\kappa + iT} \frac{\mathrm{d}s}{s} \right| \leq \frac{\kappa}{T + \kappa}.$$

Important estimates

Proof: Consider first the case when $x > 1$.

- Let k be a sufficiently large integer and let $\mathcal{R}_k$ denote the rectangle with vertices $\kappa - iT', \kappa + iT, \ \kappa - k + iT, \kappa - k - iT'$.
- Since 0 belongs to the interior of $\mathcal{R}_k$. By the Cauchy theorem, we may write

$$\frac{1}{2\pi i}\int_{\mathcal{R}_k} x^s \frac{ds}{s} = 1 = h(x).$$

- Now we have the following upper bounds

$$\left|\int_{\kappa+iT}^{\kappa-k+iT} x^s s^{-1} ds\right| \leq \int_{\kappa-k}^{\kappa} \frac{x^u du}{(u^2+T^2)^{1/2}} \leq \frac{x^\kappa}{T|\log x|},$$

$$\left|\int_{\kappa-k-iT'}^{\kappa-iT'} x^s s^{-1} ds\right| \leq \int_{\kappa-k}^{\kappa} \frac{x^u du}{(u^2+(T')^2)^{1/2}} \leq \frac{x^\kappa}{T'|\log x|},$$

$$\left|\int_{\kappa-k+iT}^{\kappa-k-iT'} x^s s^{-1} ds\right| \leq \frac{x^{\kappa-k}}{k-\kappa}\left(T+T'\right).$$

Important estimates

- The case $0 < x < 1$ can be dealt with in a symmetric way, applying the same argument with k replaced by $-k$. We omit the details.
- When $x = 1$, we simply note that

$$\frac{1}{2\pi i}\int_{\kappa-iT}^{\kappa+iT} s^{-1}ds = \frac{1}{2\pi}(\arg(\kappa+iT)-\arg(\kappa-iT)) = \frac{1}{\pi}\arctan(T/\kappa).$$

- The stated upper bound is now immediate from the following bounds

$$0 \le \frac{\pi}{2} - \arctan y = \int_y^\infty \frac{dt}{1+t^2} \le \frac{2}{1+y},$$

which is valid for all $y > 0$.
- This concludes the proof of the lemma. □

Perron truncated formula

- Let

$$F(s) := \sum_{n=1}^{\infty} b_n n^{-s},$$

be a Dirichlet series with abscissa of convergence σ_c and abscissa of absolute convergence σ_a.

Theorem (First effective Perron formula)

For $\kappa > \max\{0, \sigma_a\}$, $T \geq 1$ and $x \geq 1$, we have

$$\sum_{1 \leq n \leq x} b_n = \frac{1}{2\pi i} \int_{\kappa - iT}^{\kappa + iT} F(s) x^s \frac{ds}{s} + O\left(x^{\kappa} \sum_{n=1}^{\infty} \frac{|b_n|}{n^{\kappa}(1 + T|\log(x/n)|)} \right).$$

Perron truncated formula

- It suffices to show that, for any fixed $\kappa > 0$, and uniformly for $y > 0$, $T > 0$, we have that

$$\left|h(y) - \frac{1}{2\pi i}\int_{\kappa - iT}^{\kappa + iT} y^s s^{-1} ds\right| = O\big(y^\kappa/(1 + T|\log y|)\big). \qquad (*)$$

- Indeed, for $\kappa > \max\{0, \sigma_a\}$, $T \geq 1$ and $x \geq 1$, we have

$$\frac{1}{2\pi i}\int_{\kappa - iT}^{\kappa + iT} F(s) x^s s^{-1} ds = \sum_{n=1}^{\infty} b_n \left(\frac{1}{2\pi i}\int_{\kappa - iT}^{\kappa + iT} \left(\frac{x}{n}\right)^s \frac{ds}{s}\right).$$

- Hence, applying (*) with $y = x/n$ we obtain

$$\sum_{n=1}^{\infty} b_n \left(\frac{1}{2\pi i}\int_{\kappa - iT}^{\kappa + iT} \left(\frac{x}{n}\right)^s \frac{ds}{s}\right)$$
$$= \sum_{1 \leq n \leq x} b_n + O\left(x^\kappa \sum_{n=1}^{\infty} \frac{|b_n|}{n^\kappa (1 + T|\log(x/n)|)}\right).$$

Perron truncated formula

- It remains to prove (*):

$$\left| h(y) - \frac{1}{2\pi i} \int_{\kappa - iT}^{\kappa + iT} y^s s^{-1} ds \right| = O\big(y^\kappa / (1 + T|\log y|)\big). \qquad (*)$$

- When $T|\log y| > 1$, the estimate (*) follows from the first inequality of the previous lemma. Otherwise, when $T|\log y| \leq 1$, we can write

$$\int_{\kappa - iT}^{\kappa + iT} y^s s^{-1} ds = y^\kappa \int_{\kappa - iT}^{\kappa + iT} s^{-1} ds + y^\kappa \int_{\kappa - iT}^{\kappa + iT} \left(y^{it} - 1\right) s^{-1} ds.$$

- The second integral is

$$O\left(\int_{-T}^{T} |(t \log y) s^{-1}| dt \right) = O(T|\log y|) = O(1).$$

- Consequently, by the second inequality of the previous lemma, we see that the left-hand side of (*) is $O(y^\kappa)$ as desired. □

Perron truncated formula

Theorem (Second effective Perron formula)

Let $F(s) := \sum_{n=1}^{\infty} a_n n^{-s}$ be a Dirichlet series with $\sigma_a < \infty$.

(i) *Suppose that there exists some real number $\alpha \geq 0$ such that*

$$\sum_{n=1}^{\infty} |a_n|\, n^{-\theta} = O\big((\theta - \sigma_a)^{-\alpha} \big) \quad \text{for} \quad \theta > \sigma_a.$$

(ii) *Assume that that B is a non-decreasing function satisfying*

$$|a_n| \leq B(n) \quad \text{for all} \quad n \in \mathbb{Z}_+.$$

Then for $x \geq 2, T \geq 2, \sigma \leq \sigma_a$, and $\kappa := \sigma_a - \sigma + 1/\log x$, we have

$$\sum_{1 \leq n \leq x} \frac{a_n}{n^s} = \frac{1}{2\pi i} \int_{\kappa - iT}^{\kappa + iT} F(s+w) x^w \frac{dw}{w} + O\left(x^{\sigma_a - \sigma} \frac{(\log x)^{\alpha}}{T} + \frac{B(2x)}{x^{\sigma}} \left(1 + \frac{x \log x}{T} \right) \right).$$

Perron truncated formula

Proof: Apply the first effective Perron formula to the series $\sum_{n=1}^{\infty} b_n n^{-w}$ with $b_n := a_n n^{-s}$, we obtain that

$$\sum_{1\le n\le x} \frac{a_n}{n^s} = \frac{1}{2\pi i}\int_{\kappa-iT}^{\kappa+iT} F(s+w)x^w \frac{dw}{w} + O\left(x^{\kappa}\sum_{n=1}^{\infty}\frac{|b_n|}{n^{\kappa}(1+T|\log(x/n)|)}\right),$$

since $\kappa+\sigma > \sigma_a + 1/\log x > \sigma_a$ and $F(s+w)$ converges absolutely.

- We now write

$$\sum_{n=1}^{\infty}\frac{x^{\kappa}\,|b_n|}{n^{\kappa}(1+T|\log(x/n)|)} = \sum_{n\notin[x/2,2x]}\frac{x^{\kappa}\,|b_n|}{n^{\kappa}(1+T|\log(x/n)|)} + \sum_{n\in[x/2,2x]}\frac{x^{\kappa}\,|b_n|}{n^{\kappa}(1+T|\log(x/n)|)}.$$

Perron truncated formula

- Recalling that $|b_n| = |a_n| n^{-\sigma}$, and $\kappa = \sigma_a - \sigma + 1/\log x$, and

$$\sum_{n=1}^{\infty} |a_n|\, n^{-\theta} = O\big(\,(\theta - \sigma_a)^{-\alpha}\,\big) \quad \text{for} \quad \theta > \sigma_a,$$

  and using this bound with $\theta := \kappa + \sigma = \sigma_a + 1/\log x$, we see that the contribution of the first sum is:

$$\sum_{n \notin [x/2, 2x]} \frac{x^{\kappa} |b_n|}{n^{\kappa}(1 + T|\log(x/n)|)} = O\left(x^{\kappa} T^{-1} \sum_{n=1}^{\infty} |a_n|\, n^{-\kappa-\sigma} \right)$$
$$= O\big(x^{\sigma_a - \sigma} T^{-1} (\log x)^{\alpha}\big),$$

  since $|\log(x/n)| \geq 2$, and $x^{\kappa} = x^{\sigma_a - \sigma}$.
- For $\frac{1}{2}x \leq n \leq 2x$, using inequality $\log y \geq 1 - \frac{1}{y}$ for $y \in \mathbb{R}_+$ we have

$$|\log(x/n)| \geq \frac{|x - n|}{x}.$$

Perron truncated formula

- This leads to the following estimate

$$\sum_{n\in[x/2,2x]} \frac{x^{\kappa}\,|b_n|}{n^{\kappa}(1+T|\log(x/n)|)} = O\left(x^{-\sigma} \sum_{x/2\le n\le 2x} \frac{|a_n|}{1+T|\log(x/n)|}\right)$$

$$= O\left(\frac{B(2x)}{x^{\sigma}} \sum_{x/2\le n\le 2x} \min\left\{1, \frac{x}{T|x-n|}\right\}\right).$$

- Splitting $\sum_{x/2\le n\le 2x} = \sum_{x/2\le n\le x-1} + \sum_{x-1<n<x+1} + \sum_{x+1\le n\le 2x}$ we obtain

$$O\left(\frac{B(2x)}{x^{\sigma}} \sum_{x/2\le n\le 2x} \min\left\{1, \frac{x}{T|x-n|}\right\}\right)$$

$$= O\left(\frac{B(2x)}{x^{\sigma}}\left(1+\frac{x\log x}{T}\right)\right).$$

- This completes the proof. □

Landau's explicit formula for $\psi(x)$

Theorem (Landau)

There exists $T_0 \geq 2$ such that for any $T \geq T_0$ and for any $x \geq 2$, we have

$$\psi(x) = x - \sum_{|\operatorname{Im}\rho| \leqslant T} \frac{x^\rho}{\rho} - \log(2\pi) - \frac{1}{2}\log\left(1 - \frac{1}{x^2}\right) + O\left(\frac{x(\log xT)^2}{T} + \log x\right),$$

where

$$\psi(x) = \sum_{n \leq x} \Lambda(n).$$

Observe that for $T_0 \leq T \leq x$, we have

$$O\left(\frac{x(\log xT)^2}{T} + \log x\right) = O\left(\frac{x(\log x)^2}{T}\right).$$

Landau's explicit formula for $\psi(x)$

Proof: We may suppose that $x \notin \mathbb{Z}$.

- Recall that for every real number $T \geqslant 2$, there exists $T' \in [T, T+1]$ such that, uniformly for $-1 \leqslant \sigma \leqslant 2$, we have

$$\left|\frac{\zeta'(\sigma + iT')}{\zeta(\sigma + iT')}\right| = O(\log^2 T).$$

- Let T' be the number supplied by the above item. Let $\mathcal{R}$ be the rectangle with vertices

$$\kappa - iT', \quad \kappa + iT', \quad -(2K+1) + iT', \quad \text{and} \quad -(2K+1) - iT',$$

where $K \in \mathbb{N}$ is large.

- We know that

$$-\frac{\zeta'(s)}{\zeta(s)} = \frac{1}{s-1} + B - \sum_{n=1}^{\infty}\left(\frac{1}{2n+s} - \frac{1}{2n}\right) - \sum_{\rho}\left(\frac{1}{s-\rho} + \frac{1}{\rho}\right).$$

Landau's explicit formula for $\psi(x)$

- This implies that $-\zeta'(s)/\zeta(s)$ has simple poles at $s = -2k$ for $k \in \mathbb{Z}_+$ with residue -1, and the residue at $s = 1$, which is equal to 1.
- Therefore, by the residue theorem we obtain

$$\frac{1}{2\pi i}\int_{\mathcal{R}} -\frac{\zeta'(s)}{\zeta(s)}\frac{x^s}{s}\,\mathrm{d}s = x - \sum_{|\operatorname{Im}\rho|\leqslant T'} \frac{x^\rho}{\rho} - \sum_{1\le k<K+1/2} \frac{x^{-2k}}{-2k} - \frac{\zeta'(0)}{\zeta(0)},$$

since

$$\operatorname{res}_0\left(-\frac{\zeta'(s)}{\zeta(s)}\frac{x^s}{s}\right) = -\frac{\zeta'(0)}{\zeta(0)}, \quad \text{and} \quad \operatorname{res}_1\left(-\frac{\zeta'(s)}{\zeta(s)}\frac{x^s}{s}\right) = x,$$

$$\operatorname{res}_{-2k}\left(-\frac{\zeta'(s)}{\zeta(s)}\frac{x^s}{s}\right) = \frac{x^{-2k}}{2k}, \quad \text{and} \quad \operatorname{res}_\rho\left(-\frac{\zeta'(s)}{\zeta(s)}\frac{x^s}{s}\right) = -\frac{x^\rho}{\rho}.$$

- It can be shown that $\zeta'(0)/\zeta(0) = \log(2\pi)$.

Landau's explicit formula for $\psi(x)$

- Note that $\psi(x) = \sum_{n \le x} \Lambda(n) = \sum_{n \le x} \frac{\Lambda(n)}{n^s}$ with $s = 0$.
- We know that

$$\left| -\frac{\zeta'(1+\sigma+it)}{\zeta(1+\sigma+it)} \right| = -\frac{\zeta'(1+\sigma)}{\zeta(1+\sigma)} < \frac{1}{\sigma}$$

for any $\sigma > 0$.

- Using the second Perron formula with $\sigma_a = \alpha = 1$, $\sigma = 0 = \operatorname{Re} s$, $\kappa = 1 + 1/\log x$ and $a_n = \Lambda(n)$ and $B(n) = \log n$, we obtain

$$\begin{aligned}\psi(x) =& \frac{1}{2\pi i} \int_{\kappa - iT'}^{\kappa + iT'} \left(-\frac{\zeta'(s+w)}{\zeta(s+w)} \right) x^w \frac{dw}{w} \\ &+ O\left(\frac{x \log x}{T'} + \left(\log x + \frac{x (\log x)^2}{T'} \right) \right).\end{aligned}$$

Landau's explicit formula for $\psi(x)$

- Therefore, we may write

$$\psi(x) = x - \sum_{|\operatorname{Im}\rho| \leqslant T'} \frac{x^\rho}{\rho} - \log(2\pi) + \sum_{1 \leq k < K+1/2} \frac{x^{-2k}}{2k} - I_{\mathcal{H}_\pm} - I_{\mathcal{V}} + O\left(\frac{x(\log x)^2}{T'} + \log x\right),$$

where $I_{\mathcal{H}_\pm} = I_{\mathcal{H}_-} + I_{\mathcal{H}_+}$ and

(a) $I_{\mathcal{H}_-}$ denotes the integral taken over the bottom horizontal side connecting $-(2K+1) - iT'$ with $\kappa - iT'$,

(b) $I_{\mathcal{H}_+}$ denotes the integral taken over the top horizontal side connecting $-(2K+1) + iT'$ with $\kappa + iT'$,

(c) $I_{\mathcal{V}}$ is the integral taken over the left vertical side connecting $-(2K+1) - iT'$ with $-(2K+1) + iT'$.

Landau's explicit formula for $\psi(x)$

- Since $T' \simeq T$, we obtain

$$I_{\mathcal{H}_\pm} = O\left(\int_{-(2K+1)}^{\kappa} \left| \frac{\zeta'(\sigma \pm iT')}{\zeta(\sigma \pm iT')} \right| \frac{x^\sigma}{|\sigma \pm iT'|} d\sigma \right).$$

- We know that for $s = \sigma + it$ with $\sigma \leq -1$ one has

$$\left| \frac{\zeta'(s)}{\zeta(s)} \right| = O(\log(1 + |s|)),$$

  provided that circles of radii $1/2$ around the trivial zeros $s = -2k$ are excluded. See the corollary after the first lemma.

- Moreover, uniformly for $-1 \leqslant \sigma \leqslant 2$, we have

$$\left| \frac{\zeta'(\sigma + iT')}{\zeta(\sigma + iT')} \right| = O(\log^2 T).$$

Landau's explicit formula for $\psi(x)$

- Hence, we may conclude that

$$\begin{aligned} I_{\mathcal{H}_\pm} &= O\left(\int_{-(2K+1)}^{\kappa} \left| \frac{\zeta'(\sigma \pm iT')}{\zeta(\sigma \pm iT')} \right| \frac{x^\sigma}{|\sigma \pm iT'|} d\sigma \right) \\ &= O\left(\int_{-(2K+1)}^{-1} \frac{x^\sigma \log(1 + |\sigma \pm iT'|)}{|\sigma \pm iT'|} d\sigma + \int_{-1}^{\kappa} \frac{x^\sigma (\log T)^2}{|\sigma \pm iT'|} d\sigma \right) \\ &= O\left(\frac{x(\log T)^2}{T} \right). \end{aligned}$$

- Moreover, we have

$$\begin{aligned} I_{\mathcal{V}} &= O\left(\int_{-T'}^{T'} \left| -\frac{\zeta'(-2K-1+it)}{\zeta(-2K-1+it)} \right| \frac{x^{-2K-1}}{|-2K-1+it|} dt \right) \\ &= O\left(\frac{x^{-2K-1} T \log(KT)}{2K+1} \right). \end{aligned}$$

Landau's explicit formula for $\psi(x)$

- Letting $K \to \infty$ in the last formula we obtain

$$\begin{aligned}\psi(x) =& x - \sum_{|\operatorname{Im}\rho| \leqslant T} \frac{x^\rho}{\rho} - \log(2\pi) + \sum_{k=1}^{\infty} \frac{x^{-2k}}{2k} \\ &+ O\left(\frac{x(\log x)^2}{T} + \frac{x(\log T)^2}{T} + \log x\right),\end{aligned}$$

- This gives the desired formula, since

$$\sum_{k=1}^{\infty} \frac{x^{-2k}}{2k} = -\frac{1}{2}\log\left(1 - \frac{1}{x^2}\right).$$

- This complete the proof. □

The prime number theorem (PNT)

Theorem (PNT)

There exists an absolute constant $c \in (0,1)$ *such that as* $x \to \infty$*, one has*

$$\psi(x) = x + O\left(xe^{-c\sqrt{\log x}}\right), \tag{*}$$

$$\pi(x) = \mathrm{Li}(x) + O\left(xe^{-c\sqrt{\log x}}\right), \tag{**}$$

where

$$\mathrm{Li}(x) := \int_2^x \frac{dt}{\log t}.$$

Moreover, one has

$$\mathrm{Li}(x) = \frac{x}{\log x} + x\sum_{k=1}^{N-1} \frac{k!}{(\log x)^{k+1}} + O\left(\frac{x}{(\log x)^{N+1}}\right).$$

The prime number theorem (PNT)

Proof: For any fixed $N \in \mathbb{Z}_+$, by repeated integration by parts, we have

$$\mathrm{Li}(x) = \frac{x}{\log x} + x \sum_{k=1}^{N-1} \frac{k!}{(\log x)^{k+1}} + O\left(\frac{x}{(\log x)^{N+1}}\right).$$

- The second part (**) follows from the first part (*) by summation by parts. Therefore, it suffices to prove the first part (*).
- By Landau's theorem we obtain, for any $2 \le T \le x$, that

$$|\psi(x) - x| \le \sum_{|\operatorname{Im}\rho| \leqslant T} \frac{x^{\operatorname{Re}\rho}}{|\rho|} + O\left(\frac{x(\log x)^2}{T}\right).$$

- By the zero-free region estimates, there exists an absolute constant $C \in \mathbb{R}_+$ such that for every nontrivial zero $\rho = \beta + i\gamma$ of $\zeta(s)$, we have

$$\operatorname{Re}\rho = \beta \le 1 - \frac{C}{\log(|\gamma| + 2)} \le 1 - \frac{C}{\log T}.$$

The prime number theorem (PNT)

- Hence, inserting this bound into the previous one, we otain

$$\sum_{|\mathrm{Im}\rho|\leqslant T} \frac{x^{\mathrm{Re}\rho}}{|\rho|} \leq x^{1-\frac{c}{\log T}}(\log T)^2.$$

- Taking $T = e^{\sqrt{\log x}}$, we obtain the desired result and (*) follows. □

Corollary

As an immediate corollary, for $x \to \infty$, we obtain the following estimate

$$\pi(x) = \frac{x}{\log x} + O\left(\frac{x}{(\log x)^2}\right),$$

which is useful in many applications.

Riemann hypothesis

Riemann hypothesis (1859)

All non-trivial zeros of $\zeta(s)$ are on the critical line $\operatorname{Re} s = \frac{1}{2}$.

Theorem

The Riemann $\zeta(s) \neq 0$ *for all* $\operatorname{Re} s > 1/2$ *if and only if*

$$\psi(x) = x + O\big(\sqrt{x}(\log x)^2\big). \qquad (*)$$

Proof: ($\Longrightarrow$) For zeros $\rho = \beta + i\gamma$ of $\zeta(s)$ satisfying $|\gamma| \leq T$, we have $\beta \leq 1/2$. Choosing $T = \sqrt{x}$ and applying Landau's theorem we obtain

$$\begin{aligned} |\psi(x) - x| &\leq \sum_{|\operatorname{Im}\rho| \leqslant T} \frac{x^{\operatorname{Re}\rho}}{|\rho|} + O\left(\frac{x(\log x)^2}{T}\right) \\ &= O\left(\sqrt{x}(\log T)^2 + \sqrt{x}(\log x)^2\right) \\ &= O\left(\sqrt{x}(\log x)^2\right). \end{aligned}$$

Riemann hypothesis

We now prove the reverse implication ($\Longleftarrow$). Assume that (*) holds.

- For $\operatorname{Re} s > 1$, note that

$$-\frac{\zeta'(s)}{\zeta(s)} = \sum_{n=1}^{\infty} \frac{\Lambda(n)}{n^s} = s \int_1^{\infty} \frac{\psi(y)}{y^{s+1}} dy,$$

where $\psi(x) = \sum_{n \le x} \Lambda(n)$.

- If we set $\psi(x) = x + E(x)$, where $E(x) = O\big(\sqrt{x}(\log x)^2\big)$, then

$$-\frac{\zeta'(s)}{\zeta(s)} = \frac{s}{s-1} + s \int_1^{\infty} \frac{E(y)}{y^{s+1}} dy,$$

and clearly the integral defines a holomorphic function for $\operatorname{Re} s > 1/2$.

- Consequently the Riemann function $\zeta(s)$ cannot vanish in this region and the proof is completed. □

Argument principle

Corollary

For every real number $T \geq 2$ the number of nontrivial zeros ρ of the zeta function ζ satisfying $T \leq \operatorname{Im} \rho \leq T+1$ is at most $O(\log T)$.

Theorem

Suppose γ is a closed path in a region $\Omega \subseteq \mathbb{C}$, such that $\operatorname{Ind}_\gamma(\alpha) = 0$ for every $\alpha \notin \Omega$. Suppose also that $\operatorname{Ind}_\gamma(\alpha) = 0$ or 1 for ever $\alpha \in \Omega \setminus \gamma^*$, and let $\Omega_1 = \{\alpha \in \mathbb{C} : \operatorname{Ind}_\gamma(\alpha) = 1\}$. For any $f \in H(\Omega)$ let N_f be the number of zeros of f in Ω_1, counted according to their multiplicities. If $f \in H(\Omega)$ and f has no zeros on γ^* then

$$N_f = \frac{1}{2\pi i} \int_\gamma \frac{f'(z)}{f(z)} dz = \operatorname{Ind}_\Gamma(0),$$

where $\Gamma = f \circ \gamma$.

The Riemann–Von Mangoldt formula

Theorem

Let $N(T)$ be the number of zeros of $\zeta(s)$ with $s = \sigma + it$ in the region $0 < \sigma < 1$, $0 < t \leqslant T$. If T is not the ordinate of zero of $\zeta(s)$, then

$$N(T) = \frac{T}{2\pi} \log\left(\frac{T}{2\pi}\right) - \frac{T}{2\pi} + O(\log T) \quad \text{as} \quad T \to \infty. \qquad (*)$$

Proof: Recall that $\xi(s) = s(s-1)\pi^{-s/2}\Gamma(s/2)\zeta(s)$ and the zeros of $\xi(s)$ are nontrivial zeros of $\zeta(s)$. Moreover, $\overline{\xi(s)} = \xi(\bar{s})$.

- Let $\mathcal{R}$ denote the rectangle with vertices $2 \pm iT$, $-1 \pm iT$. Then by the argument principle we have

$$2N(T) = \frac{1}{2\pi i} \int_{\mathcal{R}} \frac{\xi'(s)}{\xi(s)} ds = \frac{1}{2\pi} \operatorname{Im} \int_{\mathcal{R}} \frac{\xi'(s)}{\xi(s)} ds,$$

since $\overline{\xi(s)} = \xi(\bar{s})$.

The Riemann–Von Mangoldt formula

- Thus we have
$$N(T) = \frac{1}{2\pi i}\int_{\mathcal{R}} \frac{\xi'(s)}{\xi(s)}ds = \frac{1}{4\pi}\,\mathrm{Im}\int_{\mathcal{R}} \frac{\xi'(s)}{\xi(s)}ds.$$
- With $\eta(s) = \pi^{-s/2}\Gamma(s/2)\zeta(s)$ we therefore may write
$$\frac{\xi'(s)}{\xi(s)} = \frac{1}{s} + \frac{1}{s-1} + \frac{\eta'(s)}{\eta(s)}.$$
- Then
$$\mathrm{Im}\left[\int_{\mathcal{R}} \left(\frac{1}{s} + \frac{1}{s-1}\right) ds\right] = 4\pi,$$
while $\eta(s) = \eta(1-s)$ and $\eta(\sigma \pm it)$ are conjugates, so that
$$\mathrm{Im}\left(\int_{\mathcal{R}} \frac{\eta'(s)}{\eta(s)}ds\right) = 4\,\mathrm{Im}\left(\int_{\mathcal{L}} \frac{\eta'(s)}{\eta(s)}ds\right),$$
where $\mathcal{L}$ consists of the segments $[2, 2+iT]$ and $\left[2+iT, \frac{1}{2}+iT\right]$.

The Riemann–Von Mangoldt formula

- Therefore

$$\operatorname{Im}\left(\int_{\mathcal{L}} \frac{\eta'(s)}{\eta(s)} ds\right) = \operatorname{Im}\left[\int_{\mathcal{L}} \left(-\frac{1}{2}\log\pi + \frac{1}{2}\frac{\Gamma'(s/2)}{\Gamma(s/2)} + \frac{\zeta'(s)}{\zeta(s)}\right) ds\right]$$
$$= -\frac{1}{2}(\log\pi)T + \operatorname{Im}\left(\int_{\mathcal{L}} \frac{\Gamma'(s/2)}{2\Gamma(s/2)} ds + \int_{\mathcal{L}} \frac{\zeta'(s)}{\zeta(s)} ds\right)$$

- Using Stirling's formula and

$$\operatorname{Im}\left(\int_{\mathcal{L}} \frac{\Gamma'(s/2)}{2\Gamma(s/2)} ds\right) = \operatorname{Im}\log\Gamma\left(\frac{1}{4} + \frac{1}{2}iT\right),$$

we obtain

$$\operatorname{Im}\left(\int_{\mathcal{L}} \frac{\Gamma'(s/2)}{2\Gamma(s/2)} ds\right) = \frac{1}{2}T\log\left(\frac{T}{2}\right) - \frac{T}{2} - \frac{\pi}{8} + O\left(\frac{1}{T}\right).$$

The Riemann–Von Mangoldt formula

- Thus from the above estimates we have

$$N(T) = \frac{T}{2\pi}\log\frac{T}{2\pi} - \frac{T}{2\pi} + \frac{7}{8} + \frac{1}{\pi}\operatorname{Im}\left(\int_{\mathcal{L}} \frac{\zeta'(s)}{\zeta(s)}ds\right) + O\left(\frac{1}{T}\right).$$

- To prove the theorem it remains to show that

$$\operatorname{Im}\left(\int_{1/2+iT}^{2+iT} \frac{\zeta'(s)}{\zeta(s)}ds\right) = O(\log T), \qquad (*)$$

since the integral over the other segment of $\mathcal{L}$ is clearly bounded.

- We also know that for $s = \sigma + it$ with $-1 \leqslant \sigma \leqslant 2$ and t not equal to an ordinate of a zero of $\zeta(s)$, we have

$$-\frac{\zeta'(s)}{\zeta(s)} = \frac{1}{s-1} - \sum_{\rho:|t-\operatorname{Im}\rho|\leqslant 1} \frac{1}{s-\rho} + O(\log(|t|+3)),$$

where the summation runs through all the non-trivial zeros of $\zeta(s)$.

The Riemann–Von Mangoldt formula

- Therefore, for $-1 \leqslant \sigma \leqslant 2$ and $t \geq 2$ not equal to an ordinate of a zero of $\zeta(s)$, we have

$$\frac{\zeta'(s)}{\zeta(s)} = \sum_{\rho:|t-\operatorname{Im}\rho|\leqslant 1} \frac{1}{s-\rho} + O(\log t),$$

where the summation runs through all the non-trivial zeros of $\zeta(s)$.

- Now the proof easily follows, since

$$\begin{aligned}\operatorname{Im}\left(\int_{1/2+iT}^{2+iT} \frac{\zeta'(s)}{\zeta(s)} ds\right) &= O(\log T) + \operatorname{Im}\left(\int_{1/2+iT}^{2+iT} \sum_{\rho:|t-\operatorname{Im}\rho|\leqslant 1} \frac{ds}{s-\rho}\right) \\ &= O(\log T) + \sum_{\rho:|t-\operatorname{Im}\rho|\leqslant 1} \Delta \arg(s-\rho) = O(\log T),\end{aligned}$$

since $|\Delta \arg(s-\rho)| < \pi$ on $\left[\frac{1}{2}+iT, 2+iT\right]$ and (*) holds. □

Borel–Carathéodory lemma

Lemma

Let $z_0 \in \mathbb{C}$, *and* $r \in (0, R)$ *and* $f(z)$ *be analytic in* $D(z_0, R)$ *given by*

$$f(z) = \sum_{n=0}^{\infty} c_n (z - z_0)^n \quad \text{for} \quad z \in D(z_0, R).$$

Let U *be a real number such that* $\mathrm{Re}(f(z)) \leq U$ *for* $z \in D(z_0, R)$. *Then*

$$|c_n| \leq \frac{2(U - \mathrm{Re}(f(z_0)))}{R^n} \quad \text{for} \quad n \in \mathbb{N}.$$

Further for $z \in \overline{D}(z_0, r)$, *we have*

$$|f(z) - f(z_0)| \leq \frac{2r}{R - r}(U - \mathrm{Re}(f(z_0))).$$

Borel–Carathéodory lemma

Proof: By considering the function $f(z+z_0)$ in place of $f(z)$, we may assume that $z_0 = 0$.

- For $z \in D(0,R)$, we write

$$f(z) = \sum_{n=0}^{\infty} c_n z^n,$$

and

$$\phi(z) = U - f(z) = U - \sum_{n=0}^{\infty} c_n z^n = \sum_{n=0}^{\infty} b_n z^n,$$

where

$$b_0 = U - c_0, \quad b_n = -c_n \quad \text{for} \quad n \in \mathbb{N} \quad \text{and} \quad \beta_0 := \mathrm{Re}\,(b_0)\,.$$

- Then for $n \geq 0$, we see that

$$b_n = \frac{1}{2\pi i} \int_{|z|=r} \frac{\phi(z)}{z^{n+1}} dz.$$

Borel–Carathéodory lemma

- Setting $z = re^{i\theta}$ with $-\pi < \theta \leq \pi$, we obtain for $n \geq 0$ that

$$b_n = \frac{1}{2\pi i}\int_{-\pi}^{\pi} \frac{\phi\left(re^{i\theta}\right) ire^{i\theta}}{r^{n+1}e^{i(n+1)\theta}} = \frac{r^{-n}}{2\pi}\int_{-\pi}^{\pi} \phi\big(re^{i\theta}\big)e^{-in\theta}d\theta.$$

- From now on, we write $\phi\big(re^{i\theta}\big) = P(r,\theta) + iQ(r,\theta) := P + iQ$, where $P(r,\theta)$ and $Q(r,\theta)$ are real-valued functions.
- Then we have

$$b_n r^n = \frac{1}{2\pi}\int_{-\pi}^{\pi}(P + iQ)e^{-in\theta}d\theta \quad \text{for} \quad n \geq 0.$$

- Next, by the Cauchy theorem, for $r < R$ and $n \geq 1$, we have

$$\begin{aligned} 0 = \frac{1}{2\pi i}\int_{|z|=r} \phi(z)z^{n-1}dz &= \frac{r^n}{2\pi i}\int_{-\pi}^{\pi} \phi\big(re^{i\theta}\big)ie^{in\theta}d\theta \\ &= \frac{r^n}{2\pi}\int_{-\pi}^{\pi}(P + iQ)e^{in\theta}d\theta. \end{aligned}$$

Borel–Carathéodory lemma

- By taking conjugates on both sides, we have

$$0 = \frac{1}{2\pi}\int_{-\pi}^{\pi}(P - iQ)e^{-in\theta}d\theta,$$

which, together with $b_n r^n = \frac{1}{2\pi}\int_{-\pi}^{\pi}(P + iQ)e^{-in\theta}d$, implies that

$$b_n r^n = \frac{1}{\pi}\int_{-\pi}^{\pi}P(r,\theta)e^{-in\theta}d\theta \quad \text{for} \quad n \in \mathbb{N}.$$

- Now we take absolute values on both sides to obtain

$$|b_n|\, r^n \le \frac{1}{\pi}\int_{-\pi}^{\pi}\left|P\left(re^{i\theta}\right)\right|d\theta \quad \text{for} \quad n \in \mathbb{N}.$$

- But we have

$$P\left(re^{i\theta}\right) = \text{Re}\left(\phi\left(re^{i\theta}\right)\right) = U - \text{Re}\left(f\left(re^{i\theta}\right)\right) \ge 0.$$

Borel–Carathéodory lemma

- Therefore
$$|b_n|\, r^n \le \frac{1}{\pi}\int_{-\pi}^{\pi} P\left(re^{i\theta}\right) d\theta \quad \text{for} \quad n \in \mathbb{N}.$$
- We recall that
$$\phi\left(re^{i\theta}\right) = \sum_{n=0}^{\infty} b_n r^n (\cos n\theta + i \sin n\theta).$$
- Therefore
$$P(r,\theta) = \mathrm{Re}\left(\phi\left(re^{i\theta}\right)\right) = \sum_{n=0}^{\infty} r^n \big(\mathrm{Re}\,(b_n) \cos n\theta - \mathrm{Im}\,(b_n) \sin n\theta\big)$$
and hence, using $\beta_0 := \mathrm{Re}\,(b_0)$, we have
$$\frac{1}{2\pi}\int_{-\pi}^{\pi} P(r,\theta) d\theta = \beta_0,$$
since $\int_{-\pi}^{\pi} \cos n\theta d\theta = 0$ for $n \ge 1$ and $\int_{-\pi}^{\pi} \sin n\theta d\theta = 0$ for $n \ge 0$.

Borel–Carathéodory lemma

- Then by $|b_n|\, r^n \leq \frac{1}{\pi}\int_{-\pi}^{\pi} P\left(re^{i\theta}\right) d\theta$ we see that

$$|b_n|\, r^n \leq 2\beta_0 \quad \text{for} \quad n \in \mathbb{N}.$$

- Letting r tend to R, we deduce that

$$|c_n| = |b_n| \leq \frac{2\beta_0}{R^n} \quad \text{for} \quad n \in \mathbb{N}.$$

- Now for $|z| \leq r < R$, we have

$$|f(z) - f(0)| = \left|\sum_{n=1}^{\infty} c_n z^n\right| \leq \sum_{n=1}^{\infty} |b_n|\, r^n \leq 2\beta_0 \sum_{n=1}^{\infty} \left(\frac{r}{R}\right)^n = 2\beta_0 \frac{r}{R-r}$$

- Inserting $\beta_0 = \mathrm{Re}\,(b_0) = U - \mathrm{Re}(f(0))$ by (5.2.5) in the above inequalities, the lemma follows. □

Nearby zeros of f'/f

Lemma

Let $f(z)$ be holomorphic for $|z-z_0| \leqslant r$, and $f(z_0) \neq 0$ and suppose that $|f(z)/f(z_0)| \leqslant M$ for $|z-z_0| \leqslant r$. If $f(z) \neq 0$ for $|z-z_0| \leqslant r/2$, and $\mathrm{Re}(z-z_0) \geqslant 0$, then

$$\mathrm{Re}\left(\frac{f'(z_0)}{f(z_0)}\right) \geqslant -\frac{4}{r}\log M \tag{*}$$

$$\mathrm{Re}\left(\frac{f'(z_0)}{f(z_0)}\right) \geqslant -\frac{4}{r}\log M + \mathrm{Re}\left(\frac{1}{z_0-\rho}\right), \tag{**}$$

where ρ is an arbitrary zero of $f(z)$ in the region $|z-z_0| \leqslant r/2$ with $\mathrm{Re}(z-z_0) < 0$.

Nearby zeros of f'/f

Proof: Let

$$g(z) = f(z) \prod_{\rho} \frac{1}{z-\rho}, \quad z \neq \rho, \quad g(\rho) = \lim_{z \to \rho} g(z),$$

where ρ denotes zeros of $f(z)$ in the circle $|z - z_0| \leqslant r/2$ counted with respective multiplicities.

- Then $g(z)$ is holomorphic for $|z - z_0| \leqslant r/2$, and for $|z - z_0| = r$

$$\left| \frac{g(z)}{g(z_0)} \right| = \left| \frac{f(z)}{f(z_0)} \prod_{\rho} \frac{z_0 - \rho}{z - \rho} \right| \leqslant M,$$

- By the maximum modulus principle this holds also for $|z - z_0| \leqslant r$.

Nearby zeros of f'/f

- Since $g(z) \neq 0$ for $|z - z_0| \leqslant r/2$, then taking the principal branch of the logarithm we see that $F(z) = \log g(z) - \log g(z_0)$ is regular for $|z - z_0| \leqslant r/2$ and

$$\operatorname{Re} F(z) = \log |g(z)/g(z_0)| \leqslant \log M,$$

and $M \geqslant 1$, since $g(z)/g(z_0) = 1$ when $z = z_0$.

- Moreover $\operatorname{Re} F(z_0) = 0$, so by the Borel–Carathéodory lemma with $R = r/2$ we obtain

$$\left|F'(z_0)\right| = \left|g'(z_0)/g(z_0)\right| \leqslant \frac{4}{r} \log M,$$

while by logarithmic differentiation we have

$$\left|\frac{g'(z_0)}{g(z_0)}\right| = \left|\frac{f'(z_0)}{f(z_0)} - \sum_{\rho} \frac{1}{z_0 - \rho}\right| \leqslant \frac{4}{r} \log M,$$

Nearby zeros of f'/f

- Thus we obtain

$$\left|\operatorname{Re}\left(\frac{f'(z_0)}{f(z_0)} - \sum_{\rho} \frac{1}{z_0 - \rho}\right)\right| \leqslant \frac{4}{r} \log M,$$

which implies

$$\operatorname{Re}\left(\frac{f'(z_0)}{f(z_0)} - \sum_{\rho} \frac{1}{z_0 - \rho}\right) \geqslant -\frac{4}{r} \log M.$$

- Finally, the condition $\operatorname{Re}(z_0 - \rho) > 0$ ensures that the conclusion of the lemma follows from the last bound, and

$$\operatorname{Re}\left(\frac{f'(z_0)}{f(z_0)}\right) \geqslant -\frac{4}{r} \log M + \operatorname{Re}\left(\frac{1}{z_0 - \rho}\right)$$

as desired. □

Zero-free region estimates revised

Theorem

Let $\varphi(t)$ *and* $1/\theta(t)$ *be two positive, nondecreasing functions of* t *for* $t \geqslant t_0$ *such that* $\theta(t) \leqslant 1$, *and* $\lim_{t\to\infty}\varphi(t) = \infty$ *and*

$$\frac{\varphi(t)}{\theta(t)} = o\left(e^{\varphi(t)}\right) \quad \text{as} \quad t \to \infty.$$

If $\zeta(s) = O\big(e^{\varphi(t)}\big)$ *for* $1 - \theta(t) \leqslant \sigma \leqslant 2$, *and* $t \geqslant t_0$, *then* $\zeta(s) \neq 0$ *for*

$$\sigma \geqslant 1 - C\frac{\theta(2t+1)}{\varphi(2t+1)} \quad \text{and} \quad t \geqslant t_0,$$

where $C > 0$ *is an absolute constant.*

Zero-free region estimates revised

Proof: Let $s = \sigma + it$. Let $\zeta(\beta + i\gamma) = 0$, with $\beta \leqslant 1$, and $\gamma \geqslant t_0$.

- Let σ_0 satisfy
$$1 + e^{-\varphi(2\gamma+1)} \leqslant \sigma_0 \leqslant 2.$$
- Let further
$$s_0 = \sigma_0 + i\gamma, \quad \text{and} \quad s_0' = \sigma_0 + 2i\gamma, \quad \text{and} \quad r = \theta(2\gamma + 1) \leq 1.$$
- Then both the circles $|s - s_0| \leqslant r$ and $|s - s_0'| \leqslant r$ lie in the region $\sigma \geqslant 1 - \theta(t)$ and $t \geqslant t_0$, since $|\sigma - \sigma_0| \leq r$, and
$$\begin{aligned}\sigma \geq \sigma_0 - r = \sigma_0 - \theta(2\gamma+1) &\geqslant 1 + e^{-\varphi(2\gamma+1)} - \theta(2\gamma+1) \\ &\geqslant 1 - \theta(2\gamma+1) \geq 1 - \frac{\theta(2\gamma+1)}{\theta(t)}\theta(t) \geq 1 - \theta(t).\end{aligned}$$
- The last inequality follows from the fact that $t \leq 2\gamma + r \leq 2\gamma + 1$, and $1/\theta(t)$ is nondecreasing. Hence $1/\theta(t) \leq 1/\theta(2\gamma+1)$ and consequently $\theta(2\gamma+1)/\theta(t) \leq 1$, giving the last lower bound.

Zero-free region estimates revised

- For $\sigma > 1$ and some $A > 0$ we have $|1/\zeta(s)| \leqslant \zeta(\sigma) < A(\sigma - 1)^{-1}$.
- Hence

$$|1/\zeta(s_0)| \leqslant Ae^{\varphi(2\gamma+1)}, \quad \text{and} \quad \left|1/\zeta\left(s_0'\right)\right| \leqslant Ae^{\varphi(2\gamma+1)},$$

  since $1 + e^{-\varphi(2\gamma+1)} \leqslant \sigma_0 \leqslant 2$.
- By hypothesis $\zeta(s) = O\big(e^{\varphi(t)}\big)$ for $1 - \theta(t) \leqslant \sigma \leqslant 2$, so that there must exist $A_2 > 0$ such that

$$\begin{aligned} |\zeta(s)/\zeta(s_0)| &< e^{A_2\varphi(2\gamma+1)} \text{ for } \quad |s - s_0| \leqslant r, \\ \left|\zeta(s)/\zeta\left(s_0'\right)\right| &< e^{A_2\varphi(2\gamma+1)} \text{ for } \quad \left|s - s_0'\right| \leqslant r. \end{aligned}$$

- Using (*) from the previous lemma with $M = e^{A_2\varphi(2\gamma+1)}$, we obtain

$$-\operatorname{Re}\frac{\zeta'(\sigma_0 + 2i\gamma)}{\zeta(\sigma_0 + 2i\gamma)} < A_3\frac{\varphi(2\gamma+1)}{\theta(2\gamma+1)} \quad \text{for some} \quad A_3 > 0. \qquad \text{(A)}$$

Zero-free region estimates revised

- We have $\beta \leqslant 1 < \sigma_0$, while for $\beta > \sigma_0 - r/2$, inequality (**) of the previous lemma gives

$$-\operatorname{Re}\frac{\zeta'(\sigma_0 + i\gamma)}{\zeta(\sigma_0 + i\gamma)} < A_3\frac{\varphi(2\gamma+1)}{\theta(2\gamma+1)} - \frac{1}{\sigma_0 - \beta}. \qquad \text{(B)}$$

- Also we have

$$-\zeta'(\sigma_0)/\zeta(\sigma_0) < B/(\sigma_0 - 1), \qquad \text{(C)}$$

where $B \to 1^+$ as $\sigma_0 \to 1^+$, since $s = 1$ is a pole of first order of $\zeta(s)$ with the residue 1.

- Recall that $3 + 4\cos\theta + \cos 2\theta = 2(1+\cos\theta)^2 \geq 0$ for any $\theta \in \mathbb{R}$.
- Hence, we have

$$-3\frac{\zeta'(\sigma_0)}{\zeta(\sigma_0)} - 4\operatorname{Re}\left(\frac{\zeta'(\sigma_0+i\gamma)}{\zeta(\sigma_0+i\gamma)}\right) - \operatorname{Re}\left(\frac{\zeta'(\sigma_0+2i\gamma)}{\zeta(\sigma_0+2i\gamma)}\right) \geq 0. \qquad \text{(D)}$$

Zero-free region estimates revised

- Inserting inequalities (A), (B) and (C) to inequality (D), we obtain

$$\frac{3B}{\sigma_0 - 1} + 5A_3 \frac{\varphi(2\gamma+1)}{\theta(2\gamma+1)} - \frac{4}{\sigma_0 - \beta} \geqslant 0,$$

  or

$$\sigma_0 - \beta \geqslant \left(\frac{3B}{4\,(\sigma_0 - 1)} + \frac{5}{4} A_3 \frac{\varphi(2\gamma+1)}{\theta(2\gamma+1)} \right)^{-1},$$

  which gives then

$$1 - \beta \geqslant \frac{1 - \frac{3}{4}B - \frac{5}{4}A_3(\varphi(2\gamma+1)/\theta(2\gamma+1))\,(\sigma_0 - 1)}{(3B/4\,(\sigma_0-1)) + \frac{5}{4}A_3(\varphi(2\gamma+1)/\theta(2\gamma+1))}.$$

- Now we choose $B = \frac{5}{4}$ and $\sigma_0 = 1 + (40A_3)^{-1}\,\theta(2\gamma+1)/\varphi(2\gamma+1)$, and then regardless of A_3 the condition $\sigma_0 \geqslant 1 + \exp(-\varphi(2\gamma+1))$ holds in view of $\frac{\varphi(t)}{\theta(t)} = o(e^{\varphi(t)})$ as $t \to \infty$.

Zero-free region estimates revised

- With this choice of B and σ_0 the last inequality reduces to

$$1-\beta \geqslant \frac{\theta(2\gamma+1)}{1240 A_3 \varphi(2\gamma+1)},$$

which gives the desired estimate provided that $\beta > \sigma_0 - r/2$.

- It remains to consider the case $\beta \leqslant \sigma_0 - r/2$, when

$$\begin{aligned}\beta \leqslant \sigma_0 - r/2 &= 1 + (40A_3)^{-1}\frac{\theta(2\gamma+1)}{\varphi(2\gamma+1)} - \frac{1}{2}\theta(2\gamma+1)\\ &\leqslant 1-(1240A_3)^{-1}\frac{\theta(2\gamma+1)}{\varphi(2\gamma+1)},\end{aligned}$$

since $\lim_{t\to\infty}\varphi(t)=\infty$. This completes the proof. □

Korobov and Vinogradov's theorem

Remark

- We can take $\theta(t) = 1/2$ and $\varphi(t) = \log(t+2)$ and by the previous theorem, we obtain that there exists an absolute constant $C > 0$ such that $\zeta(s)$ has no zero $\rho = \beta + i\gamma$ satisfying

$$\beta \geq 1 - \frac{C}{\log(|\gamma| + 2)}.$$

Theorem (Korobov and Vinogradov's theorem)

For all $s = \sigma + it \in \mathbb{C}$ such that $\frac{1}{2} \leqslant \sigma \leqslant 1$ and $t \geqslant 3$, one has

$$|\zeta(s)| \leqslant A t^{B(1-\sigma)^{3/2}} (\log t)^{2/3}. \tag{*}$$

PNT with the best error term to date

- Inequality (*) implies that there exists an absolute constant $c_0 > 0$ such that $\zeta(s)$ has no zero in the region

$$\sigma \geqslant 1 - \frac{c_0}{(\log|t|)^{2/3}(\log\log|t|)^{1/3}} \quad \text{and} \quad |t| \geqslant 3. \qquad (**)$$

- This combined with the previous theorem yields the following variant of the PNT with the best error term to date.

Theorem

There exists an absolute constant $c \in (0,1)$ such that as $x \to \infty$, one has

$$\psi(x) = x + O\left(x \exp\left(-c(\log x)^{3/5}(\log\log x)^{-1/5}\right)\right),$$

$$\pi(x) = \mathrm{Li}(x) + O\left(x \exp\left(-c(\log x)^{3/5}(\log\log x)^{-1/5}\right)\right).$$