Lecture 2

Compactification of $\mathbb{C}$
Holomorphic functions and Cauchy–Riemann equations

MATH 503, FALL 2025

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Compact sets

- Let (X, d) be a metric space and $\Omega \subseteq X$. An **open covering** of Ω is a family of open sets $(U_\alpha)_{\alpha \in A}$ (A is not necessarily countable) such that

$$\Omega \subseteq \bigcup_{\alpha \in A} U_\alpha.$$

Definition

A set Ω in a metric space is said to be compact if and only if every open covering of Ω has a finite subcovering.

Theorem

A set Ω in a metric space is compact if and only if every sequence $(z_n)_{n\in\mathbb{N}} \subseteq \Omega$ has a subsequence that converges to a point in Ω.

Theorem

A set $\Omega \subseteq \mathbb{C}$ is compact if and only if it is closed and bounded.

Compactification of $\mathbb{C}$

- By adjoining a point ∞, which we call the point at ∞, to $\mathbb{C}$ we obtain the **extended complex plane**

$$\mathbb{C}_\infty = \mathbb{C} \cup \{\infty\}.$$

- Next, we introduce the Riemann sphere

$$\mathbb{S} = \left\{(x_1, x_2, x_3) \in \mathbb{R}^3 : x_1^2 + x_2^2 + x_3^2 = 1\right\}$$

and

$$f : \mathbb{S} \to \mathbb{C}_\infty$$

given by

$$f\left((x_1, x_2, x_3)\right) = \frac{x_1 + ix_2}{1 - x_3} \quad \text{if} \quad (x_1, x_2, x_3) \neq (0, 0, 1),$$

and

$$f((0, 0, 1)) = \infty.$$

- We write $N = (0, 0, 1)$ and we call N the north pole of $\mathbb{S}$. Further the function f is called **stereographic projection**.

Stereographic projection is one-to-one

- First, we show that f is one-one. Let (x_1, x_2, x_3) and (y_1, y_2, y_3) be distinct elements of $\mathbb{S}$ such that $f((x_1, x_2, x_3)) = f((y_1, y_2, y_3))$.
- Then (x_1, x_2, x_3) and (y_1, y_2, y_3) are different from N, and therefore
$$\frac{x_1 + ix_2}{1 - x_3} = \frac{y_1 + iy_2}{1 - y_3} = z \in \mathbb{C}.$$
- Thus
$$|z|^2 = \frac{x_1^2 + x_2^2}{(1 - x_3)^2} = \frac{y_1^2 + y_2^2}{(1 - y_3)^2}.$$
- Since (x_1, x_2, x_3) and (y_1, y_2, y_3) are in $\mathbb{S}$, we have
$$|z|^2 = \frac{1 - x_3^2}{(1 - x_3)^2} = \frac{1 - y_3^2}{(1 - y_3)^2} \quad \Longleftrightarrow \quad |z|^2 = \frac{1 + x_3}{1 - x_3} = \frac{1 + y_3}{1 - y_3}.$$
- Therefore
$$\frac{|z|^2 - 1}{|z|^2 + 1} = x_3, \quad \frac{|z|^2 - 1}{|z|^2 + 1} = y_3.$$
- Thus $x_3 = y_3$ and hence $x_1 = y_1$, and $x_2 = y_2$.

Stereographic projection is onto

- Next, we show that f is onto. If $z = \infty$, we take $N \in \mathbb{S}$ so that $f(N) = \infty$.
- Thus, we may suppose that $z \in \mathbb{C}$. We take

$$x_1 = \frac{z + \overline{z}}{|z|^2 + 1}, \quad x_2 = \frac{z - \overline{z}}{i\left(|z|^2 + 1\right)}, \quad x_3 = \frac{|z|^2 - 1}{|z|^2 + 1}.$$

- Then

$$x_1^2 + x_2^2 + x_3^2 = \frac{(z + \overline{z})^2 - (z - \overline{z})^2 + \left(|z|^2 - 1\right)^2}{\left(|z|^2 + 1\right)^2}$$
$$= \frac{4|z|^2 + \left(|z|^2 - 1\right)^2}{\left(|z|^2 + 1\right)^2} = 1$$

implying $(x_1, x_2, x_3) \in \mathbb{S}$. Further, we check that $f\left((x_1, x_2, x_3)\right) = z$.

Geometric interpretation of the stereographic projection

- Let $z = x + iy$ be a point in the complex plane and we identify it by $(x, y, 0)$ in $\mathbb{R}^3$. We consider the line joining z to N in $\mathbb{R}^3$.
- Its parametric representation is

$$tN + (1 - t)z, \quad \text{for} \quad -\infty < t < \infty.$$

- The points on this line are

$$((1 - t)x, (1 - t)y, t), \quad \text{for} \quad -\infty < t < \infty.$$

- Thus, the line intersects $\mathbb{S}$ if and only if

$$(1 - t)^2x^2 + (1 - t)^2y^2 + t^2 = 1,$$

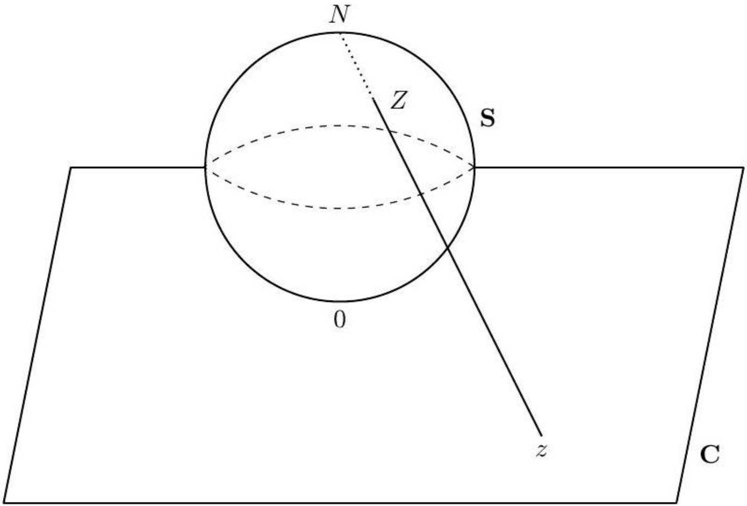
which we re-write as

$$(1 - t)|z|^2 = (1 - t)x^2 + (1 - t)y^2 = 1 + t.$$

- Hence

$$t = \frac{|z|^2 - 1}{|z|^2 + 1}.$$

Geometric interpretation of the stereographic projection



Geometric interpretation of the stereographic projection

- Thus, the line intersects $\mathbb{S}$ at Z given by

$$Z = \left(\left(1 - \frac{|z|^2 - 1}{|z|^2 + 1}\right) x, \left(1 - \frac{|z|^2 - 1}{|z|^2 + 1}\right) y, \frac{|z|^2 - 1}{|z|^2 + 1} \right)$$
$$= \left(\frac{2x}{|z|^2 + 1}, \frac{2y}{|z|^2 + 1}, \frac{|z|^2 - 1}{|z|^2 + 1} \right)$$
$$= \left(\frac{z + \bar{z}}{|z|^2 + 1}, \frac{z - \bar{z}}{i\,(|z|^2 + 1)}, \frac{|z|^2 - 1}{|z|^2 + 1} \right).$$

- We summarize the above procedure as follows:
  - Let $z = (x, y)$ be a point in the complex plane.
  - The line passing through $(x, y, 0)$ and $(0, 0, 1)$ intersects $\mathbb{S}$ exactly at one point $Z = (x_1, x_2, x_3)$ given above and $f\left((x_1, x_2, x_3)\right) = z$.
  - We call (x_1, x_2, x_3) the spherical coordinates of z and $(0, 0, 1)$ the spherical coordinates of ∞.

Distance between points in the extended complex plane

- Let $z \in \mathbb{C}_\infty$ and $z' \in \mathbb{C}_\infty$. Let (x_1, x_2, x_3) and (x_1', x_2', x_3') be spherical coordinates of z and z', respectively. Namely,

$$x_1 = \frac{z + \overline{z}}{|z|^2 + 1}, \quad x_2 = \frac{z - \overline{z}}{i\left(|z|^2 + 1\right)}, \quad x_3 = \frac{|z|^2 - 1}{|z|^2 + 1},$$

and

$$x_1' = \frac{z' + \overline{z}'}{|z'|^2 + 1}, \quad x_2' = \frac{z' - \overline{z}'}{i\left(|z'|^2 + 1\right)}, \quad x_3' = \frac{|z'|^2 - 1}{|z'|^2 + 1}.$$

- Then we define

$$d\left(z, z'\right) = \sqrt{\left(x_1 - x_1'\right)^2 + \left(x_2 - x_2'\right)^2 + \left(x_3 - x_3'\right)^2}.$$

- Thus

$$\begin{aligned} d\left(z, z'\right)^2 &= \left(x_1 - x_1'\right)^2 + \left(x_2 - x_2'\right)^2 + \left(x_3 - x_3'\right)^2 \\ &= 2 - 2\left(x_1 x_1' + x_2 x_2' + x_3 x_3'\right). \end{aligned}$$

Distance between points in the extended complex plane

- Now we note

$$x_1x_1' + x_2x_2' + x_3x_3'$$

$$= \frac{(z+\overline{z})(z'+\overline{z}')}{(|z|^2+1)\left(|z'|^2+1\right)} - \frac{(z-\overline{z})(z'-\overline{z}')}{(|z|^2+1)\left(|z'|^2+1\right)} + \frac{(|z|^2-1)\left(|z'|^2-1\right)}{(|z|^2+1)\left(|z'|^2+1\right)}$$

$$= \frac{(|z|^2-1)\left(|z'|^2-1\right) + 2z\overline{z}' + 2\overline{z}z'}{(|z|^2+1)\left(|z'|^2+1\right)}$$

$$= \frac{(|z|^2+1)\left(|z'|^2+1\right) - 2\left(|z|^2+1\right) - 2\left(|z'|^2+1\right) + 4 + 2z\overline{z}' + 2\overline{z}z'}{(|z|^2+1)\left(|z'|^2+1\right)}$$

- Further

$$2\left(|z|^2+1\right) + 2\left(|z'|^2+1\right) - 4 - 2z\overline{z}' - 2\overline{z}z'$$

$$= 2\left(|z|^2 + |z'|^2 - z\overline{z}' - \overline{z}z'\right) = 2\left|z-z'\right|^2.$$

Distance between points in the extended complex plane

- Hence
$$2\left(x_1x_1' + x_2x_2' + x_3x_3'\right) = 2 - \frac{4\left|z - z'\right|^2}{\left(|z|^2+1\right)\left(\left|z'\right|^2+1\right)}.$$
- Since $d\left(z, z'\right)^2 = 2 - 2\left(x_1x_1' + x_2x_2' + x_3x_3'\right)$, we obtain
$$d\left(z, z'\right)^2 = \frac{4\left|z - z'\right|^2}{\left(|z|^2+1\right)\left(\left|z'\right|^2+1\right)}$$
- Hence
$$d\left(z, z'\right) = \frac{2\left|z - z'\right|}{\sqrt{\left(|z|^2+1\right)\left(\left|z'\right|^2+1\right)}}$$
- If $z' = \infty$. Then $x_1' = 0, x_2' = 0$ and $x_3' = 1$ and
$$d(z,\infty)^2 = 2 - 2x_3 = 2 - 2\left(\frac{|z|^2-1}{|z|^2+1}\right) = \frac{4}{|z|^2+1}.$$

Distance between points in the extended complex plane

- Hence
$$d(z,\infty) = \frac{2}{\sqrt{|z|^2+1}}.$$
- Thus $\mathbb{C}_\infty$ is a metric space.
- The above metric d induces the following topology on $\mathbb{C}_\infty$:
  - Let $E \subseteq \mathbb{C}_\infty$.
  - If $\infty \notin E$, then E is open in $\mathbb{C}_\infty$ if and only if it is open in $\mathbb{C}$.
  - If $\infty \in E$, then E is open in $\mathbb{C}_\infty$ if and only its complement in $\mathbb{C}_\infty$ is a closed and bounded (i.e. compact) subset of $\mathbb{C}$.
- Finally, we check that f and f^{-1} are continuous to conclude the following result.

Theorem

The extended complex plane $\mathbb{C}_\infty$ is a metric space homeomorphic to the Riemann sphere $\mathbb{S}$. In particular, $\mathbb{C}_\infty$ is compact.

Functions on the complex plane

- Let $E \subseteq \mathbb{C}$ be a set of complex numbers. A function f defined on E is a rule that assigns to each z in E a **unique complex number** w.
- The number w is called the value of f at z and is denoted by $f(z)$. The set E is called the domain of definition of f.
- Suppose $w = u + iv$ is the value of a function f at $z = x + iy$, meaning
$$u + iv = f(x + iy).$$
- The real numbers u and v depend on the real variables x and y.
- Consequently, $f(z)$ can be expressed as a pair of real-valued functions of the real variables x and y as follows:
$$f(z) = u(x, y) + iv(x, y).$$
- The functions u and v are usually denoted by $\operatorname{Re} f(z)$ and $\operatorname{Im} f(z)$ respectively.

Examples

- Consider the function $f(z) = z^2$. Then $f(x+iy) = x^2 - y^2 + i2xy$. Therefore, we have $\operatorname{Re} f(z) = x^2 - y^2$ and $\operatorname{Im} f(z) = 2xy$. The domain of f is the entire complex plane.
- Let us consider $f(z) = z^{1/n}$, where we assign to each complex number its n-th root. Then f is not a function in the sense of the definition above, since for each $z \neq 0$, there are n roots of z.
- Let us express z as $z = |z|(\cos\theta + i\sin\theta)$, where $\theta \in (-\pi, \pi]$ is the argument of z. Then the roots of z are given by

$$z_k = \sqrt[n]{|z|}\left(\cos\frac{\theta + 2k\pi}{n} + i\sin\frac{\theta + 2k\pi}{n}\right), \quad k = 0, 1, \ldots, n-1.$$

- Clearly, for each $z \in \mathbb{C}$, we have n values of $z^{1/n}$ determined by k. In this case, we say that the n-th root has n **branches**. We can make $z^{1/n}$ a single-valued function by choosing a specific k.
- A branch corresponding to $k = 0$ is called the **principal branch**. Each branch is a single-valued function.

Continuity of complex functions

Definition

Let $E \subseteq \mathbb{C}$ and let $f : E \to \mathbb{C}$ be a function. We say that f is continuous at the point $z_0 \in E$ if for every $\varepsilon > 0$ there exists $\delta > 0$ such that whenever $z \in E$ and $|z - z_0| < \delta$ then

$$|f(z) - f(z_0)| < \varepsilon.$$

An equivalent definition is that for every sequence $(z_n)_{n\in\mathbb{N}} \subset E$ such that $\lim_{n\to\infty} z_n = z_0$, then

$$\lim_{n\to\infty} f(z_n) = f(z_0).$$

- The function f is said to be continuous on E if it is continuous at every point of E.
- Sums and products of continuous functions are also continuous.
- The function f is continuous if and only if its real and imaginary parts are continuous.

Properties of continues functions

- Since the convergence notions for the complex numbers and points in $\mathbb{R}^2$ are the same, the function f of the complex argument $z = x + iy$ is continuous if and only if it is continuous when viewed as a function of the two real variables x and y.
- By the triangle inequality, it is immediate that if f is continuous, then the real-valued function defined by $z \mapsto |f(z)|$ is continuous.
- We say that f attains a maximum at the point $z_0 \in E$ if

$$|f(z)| \leq |f(z_0)| \quad \text{for all} \quad z \in E,$$

with the inequality reversed for the definition of a minimum.

Theorem

A continuous complex function on a compact set E is bounded and attains a maximum and minimum on E.

Examples

- The function $f(z) = z^2$ is continuous on $\mathbb{C}$, which can be easily shown using the definition of continuity.
- The principal branch of the square root function $z^{1/2}$ is continuous on $\mathbb{C} \setminus (-\infty, 0]$.
- Recall that

$$z_k = \sqrt{|z|}\left(\cos\frac{\theta + 2k\pi}{2} + i\sin\frac{\theta + 2k\pi}{2}\right), \quad k = 0, 1.$$

- This exclusion is necessary because for $z = x + iy$ where $x \leq 0$ and $y \to 0$ we may observe the following:
  - If $y \to 0^+$, then $\arg(z) \to \pi$, leading to $z_0 \to i\sqrt{|x|}$.
  - If $y \to 0^-$, then $\arg(z) \to -\pi$, leading to $z_0 \to -i\sqrt{|x|}$.
- We see that the principal branch $z^{1/2}$ exhibits discontinuity depending on the direction of approach along the imaginary axis.

Complex Differentiation

Definition

Suppose f is a complex function defined in open set $\Omega \subseteq \mathbb{C}$. If $z_0 \in \Omega$ and

$$\lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0}$$

exists, we denote this limit by $f'(z_0)$ and call it the derivative of f at z_0.

- If the derivative $f'(z_0)$ exists for every $z_0 \in \Omega$, we say that f is **holomorphic** (or **analytic**) in Ω.
- The class of all holomorphic functions in Ω will be denoted by $H(\Omega)$
- To be more explicit, the derivative $f'(z_0)$ exists if to every $\varepsilon > 0$ there exists a $\delta > 0$ such that

$$\left| \frac{f(z) - f(z_0)}{z - z_0} - f'(z_0) \right| < \varepsilon \quad \text{for all} \quad z \in D'(z_0, \delta).$$

- A function holomorphic in the whole $\mathbb{C}$ is called an **entire** function.

Properties of holomorphic functions

- If $f \in H(\Omega)$ and $g \in H(\Omega)$, then $f + g \in H(\Omega)$ and $fg \in H(\Omega)$, so that $H(\Omega)$ is a ring.
- Moreover, if $f \in H(\Omega)$ and $g \in H(\Omega)$ and additionally g is never 0 on Ω, then $f/g \in H(\Omega)$.
- The usual differentiation rules apply: for any $f \in H(\Omega)$ and $g \in H(\Omega)$ and $\alpha, \beta \in \mathbb{C}$ we have

$$\begin{aligned} (\alpha f + \beta g)' &= \alpha f' + \beta g', \\ (fg)' &= f'g + fg', \end{aligned}$$

- If $f \in H(\Omega)$ and $g \in H(\Omega)$ and additionally g is never 0 on Ω, then

$$\left(\frac{f}{g}\right)' = \frac{f'g - fg'}{g^2}.$$

- More interestingly, superpositions of holomorphic functions are also holomorphic.

Properties of holomorphic functions

Lemma

If $f \in H(\Omega)$, $f(\Omega) \subseteq \Omega_1$, $g \in H(\Omega_1)$, and $h = g \circ f$, then $h \in H(\Omega)$, and h' can be computed using the chain rule:

$$h'(z_0) = g'(f(z_0)) \cdot f'(z_0), \quad \textit{for} \quad z_0 \in \Omega.$$

Proof: To prove this, fix $z_0 \in \Omega$, and put $w_0 = f(z_0)$. Then

$$f(z) - f(z_0) = \left[f'(z_0) + \varepsilon(z)\right](z - z_0),$$
$$g(w) - g(w_0) = \left[g'(w_0) + \eta(w)\right](w - w_0),$$

where $\lim_{z \to z_0} \varepsilon(z) = 0$ and $\lim_{w \to w_0} \eta(w) = 0$. Put $w = f(z)$, and substitute it to the last equation. If $z \neq z_0$, we obtain

$$\frac{h(z) - h(z_0)}{z - z_0} = \left[g'(f(z_0)) + \eta(f(z))\right]\left[f'(z_0) + \varepsilon(z)\right].$$

The differentiability of f forces f to be continuous at z_0. Hence the conclusion follows from the last equation. □

Properties of holomorphic functions

Theorem

Let g be analytic on the open set Ω_1, and let f be a continuous complex-valued function on the open set Ω. Assume that

(i) $f(\Omega) \subseteq \Omega_1$,

(ii) g' *is never 0* ,

(iii) $g(f(z)) = z$ *for all* $z \in \Omega$ *(thus f is one-to-one).*

Then f is analytic on Ω and $f' = 1/(g' \circ f)$.

Proof: Let $z_0 \in \Omega$, and let $(z_n)_{n\in\mathbb{N}} \subseteq \Omega \setminus \{z_0\}$ with $\lim_{n\to\infty} z_n = z_0$. Then

$$\frac{f(z_n) - f(z_0)}{z_n - z_0} = \frac{f(z_n) - f(z_0)}{g(f(z_n)) - g(f(z_0))} = \left[\frac{g(f(z_n)) - g(f(z_0))}{f(z_n) - f(z_0)}\right]^{-1}$$

(Note that $f(z_n) \neq f(z_0)$ since f is 1-1 and $z_n \neq z_0$.) By continuity of f at z_0, the expression in brackets approaches $g'(f(z_0))$ as $n \to \infty$. Since $g'(f(z_0)) \neq 0$, the result follows. □

Examples

- The function $f(z) = z$ is holomorphic on any open set in $\mathbb{C}$, and $f'(z) = 1$. In fact, any polynomial

$$p(z) = a_0 + a_1 z + \cdots + a_n z^n$$

is holomorphic in the entire complex plane and

$$p'(z) = a_1 + 2a_2 z + \cdots + n a_n z^{n-1}.$$

- The function $\frac{1}{z}$ is holomorphic on any open set in $\mathbb{C}$ that does not contain the origin, and $f'(z) = -\frac{1}{z^2}$.
- The function $f(z) = \overline{z}$ is not holomorphic. Indeed, we have

$$\frac{f(z_0 + h) - f(z_0)}{h} = \frac{\overline{h}}{h}$$

which has no limit as $h \to 0$, as one can see by first taking h real and then h purely imaginary.

Real differentiability

Definition

Suppose that $E \subseteq \mathbb{R}^n$ is open $F : E \to \mathbb{R}^m$, and $x \in E$. If there exists a linear transformation $A : \mathbb{R}^n \to \mathbb{R}^m$ such that

$$\lim_{h\to 0} \frac{|F(x+h) - F(x) - Ah|}{|h|} = 0, \qquad (*)$$

then we say that F is **differentiable** at x, and we write

$$A = F'(x).$$

If F is differentiable at every $x \in E$, we say that F is differentiable in E.

- The relation (*) can be rewritten in the form

$$F(x+h) - F(x) = F'(x)h + r(h), \quad \text{and} \quad \lim_{h\to 0} \frac{|r(h)|}{|h|} = 0. \qquad (**)$$

Real differentiability

- If $F = (F_1, \dots, F_m) : E \to \mathbb{R}^m$ and $F'(x_0)$ exists at $x_0 \in E$, then

$$F'(x_0) = \begin{bmatrix} \frac{\partial F_1}{\partial x_1}(x_0) & \dots & \frac{\partial F_1}{\partial x_n}(x_0) \\ \vdots & \ddots & \vdots \\ \frac{\partial F_m}{\partial x_1}(x_0) & \dots & \frac{\partial F_m}{\partial x_n}(x_0) \end{bmatrix}.$$

- We know that if all the partial derivatives $\frac{\partial F_1}{\partial x_1}, \dots, \frac{\partial F_i}{\partial x_j}, \dots, \frac{\partial F_m}{\partial x_n}$ of F exist in a neighborhood of $x_0 \in E$ and they are continues at the point x_0, then the function F is differentiable at x_0.
- The converse is not true. Consider the example

$$F(x, y) = \begin{cases} \frac{y^3}{x^2+y^2} & \text{if } (x, y) \neq (0, 0), \\ 0 & \text{if } (x, y) = (0, 0). \end{cases}$$

  The function F is not differentiable at the point $(0, 0)$, even though the partial derivatives $\frac{\partial F}{\partial x}$ and $\frac{\partial F}{\partial y}$ exist at this point. However, these partial derivatives are not continuous at the origin.

Real differentiability

- Recall that the function $f(z) = \overline{z}$ is not holomorphic. Indeed, we have
$$\frac{f(z_0 + h) - f(z_0)}{h} = \frac{\overline{h}}{h},$$
which has no limit as $h \to 0$, as one can see by first taking h real and then h purely imaginary.
- If the function $f(z) = \overline{z}$ is considered as a real variable function then it corresponds to the map $F(x, y) = (x, -y)$, which is indefinitely differentiable in the real sense.
- This example illustrates that the existence of the real derivative need not guarantee that f is holomorphic.
- This example leads us to associate more generally to each complex valued function
$$f = u + iv,$$
the mapping from $F : E \to \mathbb{R}^2$, given by
$$F(x, y) = (u(x, y), v(x, y)).$$

$F(x,y) = (u(x,y), v(x,y))$

- If $m = n = 2$ and $(x_0, y_0) \in E$ is fixed and $h = (h_1, h_2)$ and $|h| \to 0$, then the relation (**) can be rewritten in the form

$$u(x_0 + h_1, y_0 + h_2) - u(x_0, y_0) = \frac{\partial u}{\partial x}(x_0, y_0)h_1 + \frac{\partial u}{\partial y}(x_0, y_0)h_2 + r_1(h_1, h_2),$$

and

$$v(x_0 + h_1, y_0 + h_2) - v(x_0, y_0) = \frac{\partial v}{\partial x}(x_0, y_0)h_1 + \frac{\partial v}{\partial y}(x_0, y_0)h_2 + r_2(h_1, h_2),$$

where $r(h_1, h_2) = (r_1(h_1, h_2), r_2(h_1, h_2))$ and

$$\lim_{h \to 0} \frac{|r_1(h_1, h_2)|}{|h|} = 0, \quad \text{and} \quad \lim_{h \to 0} \frac{|r_2(h_1, h_2)|}{|h|} = 0.$$

Cauchy–Riemann equations

Theorem

Let $\Omega \subseteq \mathbb{C}$ be open and $f : \Omega \to \mathbb{C}$ be holomorphic. Then

$$\frac{\partial f}{\partial x} = \frac{1}{i}\frac{\partial f}{\partial y}, \tag{A}$$

where $\partial/\partial x$ and $\partial/\partial y$ denote the usual partial derivatives in the x and y variables respectively. If $f = u + iv$ for some real valued functions $u, v : \Omega \to \mathbb{C}$, then we have

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad \text{and} \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}. \tag{B}$$

These relations are called the **Cauchy–Riemann** *equations.*

Cauchy–Riemann equations

Proof: Fix $z_0 \in \Omega$. Since $f : \Omega \to \mathbb{C}$ is holomorphic, we have

$$\lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0} = f'(z_0). \tag{C}$$

- To find relation (A), observe that $z_0 = (x_0, y_0)$ can be approached from different directions.
- Firstly, with z of the form $(x_0 + h) + iy_0$ in (C), observe that

$$f'(z_0) = \lim_{h \to 0} \frac{f(x_0 + h, y_0) - f(x_0, y_0)}{h} = \frac{\partial f}{\partial x}.$$

- Secondly, with z of the form $x_0 + i(y_0 + h)$ in (C), observe that

$$f'(z_0) = \lim_{h \to 0} \frac{f(x_0, y_0 + h) - f(x_0, y_0)}{ih} = \frac{1}{i}\frac{\partial f}{\partial y}.$$

Cauchy–Riemann equations

- Thus we obtain equation (A):

$$\frac{\partial f}{\partial x} = \frac{1}{i}\frac{\partial f}{\partial y}.$$

- If $f = u + iv$ for some real valued functions $u, v : \Omega \to \mathbb{C}$, then

$$\frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} = -i\frac{\partial u}{\partial y} + \frac{\partial v}{\partial y}.$$

- This implies equations in (B):

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad \text{and} \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}.$$

- This completes the proof. □

Cauchy–Riemann equations and holomorphicity

Theorem

Suppose $f = u + iv$ is a complex-valued function defined on an open set Ω. If u and v are differentiable in the real sense and satisfy the Cauchy–Riemann equations on Ω, then f is holomorphic on Ω.

Proof: Fix $z_0 = x_0 + iy_0 \in \Omega$ and let $h = h_1 + ih_2$. Recall that

$$u(x_0 + h_1, y_0 + h_2) - u(x_0, y_0) = \frac{\partial u}{\partial x}(x_0, y_0)h_1 + \frac{\partial u}{\partial y}(x_0, y_0)h_2 + r_1(h_1, h_2),$$

and

$$v(x_0 + h_1, y_0 + h_2) - v(x_0, y_0) = \frac{\partial v}{\partial x}(x_0, y_0)h_1 + \frac{\partial v}{\partial y}(x_0, y_0)h_2 + r_2(h_1, h_2),$$

where

$$\lim_{h \to 0} \frac{|r_1(h_1, h_2)|}{|h|} = 0, \quad \text{and} \quad \lim_{h \to 0} \frac{|r_2(h_1, h_2)|}{|h|} = 0.$$

Cauchy–Riemann equations and holomorphicity

- Thus

$$
\begin{aligned}
f(z_0+h)-f(z_0) =& \frac{\partial u}{\partial x}(x_0,y_0)h_1 + \frac{\partial u}{\partial y}(x_0,y_0)h_2 \\
& + i\left(\frac{\partial v}{\partial x}(x_0,y_0)h_1 + \frac{\partial v}{\partial y}(x_0,y_0)h_2\right) + r(h),
\end{aligned}
$$

where $r(h) = r_1(h_1,h_2) + ir_2(h_1,h_2)$. Clearly $\lim_{h\to 0} \frac{|r(h)|}{|h|} = 0$.

- By the Cauchy–Riemann equations $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$, $\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$, we find that

$$
f(z_0+h)-f(z_0) = \left(\frac{\partial u}{\partial x}(x_0,y_0) - i\frac{\partial u}{\partial y}(x_0,y_0)\right)h + r(h).
$$

- Then

$$
f'(z_0) = \frac{\partial u}{\partial x}(x_0,y_0) - i\frac{\partial u}{\partial y}(x_0,y_0). \qquad \square
$$

Example

- Note that the hypothesis of real differentiability at the point z_0 is essential and cannot be dispensed with.
- For example, consider the function

$$f(x, y) = \sqrt{|xy|},$$

  regarded as a complex function with imaginary part identically zero.
    - Then it has both partial derivatives at $(x_0, y_0) = (0, 0)$.
    - It moreover satisfies the Cauchy–Riemann equations at that point.
    - But it is not differentiable in the real sense, and so the first condition, that of real differentiability, is not met.
- Therefore, this function is not complex differentiable.