Lecture 19

Dirichlet series

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Arithmetic functions

Definition

An **arithmetic function** is a map $f : \mathbb{Z}_+ \to \mathbb{C}$, i.e., a sequence of complex numbers, although this viewpoint is not very useful.

Examples of arithmetic functions

- The **constant 1** and the **identity** Id functions are defined by

$$\mathbf{1}(n) := 1 \qquad \text{and} \qquad \mathrm{Id}(n) := n \quad \text{for all} \quad n \in \mathbb{Z}_+.$$

- The **Dirac delta** function δ_m is defined as follows

$$\delta_m(n) = \begin{cases} 1 & \text{if } n = m, \\ 0 & \text{otherwise.} \end{cases}$$

We shall abbreviate δ_1 to δ.

Examples of arithmetic functions

- The **divisor** function $\tau(n)$ is the number of positive divisors of $n \in \mathbb{Z}_+$,
$$\tau(n) := \#\{d \in \mathbb{Z}_+ : d \mid n\} = \sum_{d|n} 1.$$
Some authors also use the notation $d(n)$ for the divisor function.
- More generally, the **sum of powers of divisors** is defined by
$$\sigma_k(n) := \sum_{d|n} d^k, \quad \text{where} \quad k \in \mathbb{N}.$$
Observe that $\tau(n) = \sigma_0(n)$, and we abbreviate σ_1 to σ.
- The **Euler totient function** φ is defined by
$$\varphi(n) := \#\{m \in [n] : (n, m) = 1\} = \sum_{m \in [n]} \delta((n, m)).$$
Here and throughout, we use the convention from combinatorics that $[N] := (0, N] \cap \mathbb{Z}_+$ for any real number $N > 0$.

Examples of arithmetic functions

- The **function** ω is defined as follows: $\omega(1) = 0$ and $\omega(n)$ counts the number of distinct prime factors of n for all $n \geq 2$.
- The **function** Ω is defined as follows: $\Omega(1) = 0$ and $\Omega(n)$ counts the number of prime factors of n with multiplicities for all $n \geq 2$.
- The **Liouville function** λ is defined as follows

$$\lambda(n) = (-1)^{\Omega(n)}.$$

- The **Möbius function** $\mu(n)$ is defined as follows

$$\mu(n) := \begin{cases} 1 & \text{if } n = 1, \\ (-1)^k & \text{if } n \text{ is the product of } k \text{ distinct primes,} \\ 0 & \text{if } n \text{ is divisible by the square of a prime.} \end{cases}$$

Dirichlet convolutions

- The **von Mangoldt function** $\Lambda(n)$ is defined as follows

$$\Lambda(n) := \begin{cases} \log p & \text{if } n = p^k \text{ is a prime power,} \\ 0 & \text{otherwise.} \end{cases}$$

- Sums and products of arithmetic functions are arithmetic functions:

$$(f+g)(n) := f(n) + g(n) \quad \text{and} \quad (f \cdot g)(n) := f(n) \cdot g(n).$$

Definition

The **Dirichlet convolution** $f \star g$ is defined by

$$(f \star g)(n) = \sum_{d|n} f(d) g\left(\frac{n}{d}\right),$$

where the sum is over all positive divisors d of n. Dirichlet convolution occurs frequently in multiplicative problems in elementary number theory.

Ring of arithmetic functions

Theorem

The set $\mathbb{A} := (\mathbb{A}, +, \star)$ *of all complex-valued arithmetic functions, with addition* $+$ *defined by pointwise sum and multiplication* $\star$ *defined by Dirichlet convolution, is a commutative ring with additive identity* 0 *and multiplicative identity* δ*, which is the Dirac delta at* 1*. Furthermore, if* $f(1) \neq 0$*, then* f *is invertible.*

Proof: Prove it!

Multiplicative functions

Definition

Let $f : \mathbb{Z}_+ \to \mathbb{C}$ be an arithmetic function.

- The function f is said to be **multiplicative** if $f(1) \neq 0$ and if, for all positive integers $m, n \in \mathbb{Z}_+$ such that $(m, n) = 1$, we have
$$f(mn) = f(m)f(n).$$
- The function f is **completely multiplicative** if $f(1) \neq 0$ and if the condition
$$f(mn) = f(m)f(n)$$
holds for all positive integers m and n.
- The function f is **strongly multiplicative** if f is multiplicative and if $f(p^\alpha) = f(p)$ for all prime powers p^α.

Multiplicative functions

Remark

- The condition $f(1) \neq 0$ is a convention to exclude the zero function from the set of multiplicative functions.
- Furthermore, it is easily seen that if f and g are multiplicative, then so are fg and f/g with $g \neq 0$ for the quotient.

Additive functions

Definition

Let $f : \mathbb{Z}_+ \to \mathbb{C}$ be an arithmetic function.

- The function f is said to be **additive** if for all positive integers $m, n \in \mathbb{Z}_+$ such that $(m, n) = 1$, we have

$$f(mn) = f(m) + f(n).$$

- The function f is **completely additive** if the condition

$$f(mn) = f(m) + f(n)$$

  holds for all positive integers m and n.
- The function f is **strongly additive** if f is multiplicative and if $f(p^\alpha) = f(p)$ for all prime powers p^α.

Additive and multiplicative functions: simple criterium

Lemma

Let $f : \mathbb{Z}_+ \to \mathbb{C}$ be an arithmetic function.

(i) *f is multiplicative if and only if $f(1) = 1$ and for all $n = p_1^{\alpha_1} \cdots p_r^{\alpha_r}$, where the p_i are distinct primes, we have*

$$f(n) = \prod_{j \in [r]} f\left(p_j^{\alpha_j}\right).$$

(ii) *f is additive if and only if $f(1) = 0$ and for all $n = p_1^{\alpha_1} \cdots p_r^{\alpha_r}$, where the p_i are distinct primes, we have*

$$f(n) = \sum_{j \in [r]} f\left(p_j^{\alpha_j}\right).$$

Proof: Prove it!

Proof

Theorem

If $f, g : \mathbb{Z}_+ \to \mathbb{C}$ are multiplicative, then so is $f \star g$.

Proof: Let f and g be two multiplicative functions and let $m, n \in \mathbb{Z}_+$ be such that $(m, n) = 1$.

- Note that each divisor d of mn can be written uniquely in the form $d = ab$ with $a \mid m$, and $b \mid n$ and $(a, b) = 1$.
- Hence,

$$(f \star g)(mn) = \sum_{d|mn} f(d)g\left(\frac{mn}{d}\right) = \sum_{a|m}\sum_{b|n} f(ab)g\left(\frac{mn}{ab}\right).$$

- Since f and g are multiplicative and $(a, b) = (m/a, n/b) = 1$, then

$$(f \star g)(mn) = \sum_{a|m}\sum_{b|n} f(a)f(b)g\left(\frac{m}{a}\right)g\left(\frac{n}{b}\right) = (f \star g)(m)(f \star g)(n)$$

as required. □

Examples

- The functions $\mathbf{1}$, Id, δ are completely multiplicative.
- The functions log and Ω are strongly additive and consequently the function λ is completely multiplicative.
- Since $\tau = \mathbf{1} \star \mathbf{1}$, $\sigma = \mathbf{1} \star \mathrm{Id}$ and $\sigma_k = \mathbf{1} \star \mathrm{Id}^k$ and both $\mathbf{1}$ and Id are completely multiplicative, so are d, σ and σ_k.
- It is easily seen that, for all $m, n \in \mathbb{Z}_+$, we have

  $$\omega(mn) = \omega(m) + \omega(n) - \omega(m, n),$$

  since in the sum $\omega(m) + \omega(n)$, the prime factors of (m, n) have been counted twice. This implies the additivity of ω.

Möbius function is multiplicative

- The Möbius function $\mu(n)$ is multiplicative. Indeed, $\mu(1) = 1$ and for all prime powers p^α, we also have

$$\mu(p^\alpha) = \begin{cases} -1, & \text{if } \alpha = 1, \\ 0, & \text{otherwise.} \end{cases}$$

- So that, if $n = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_r^{\alpha_r}$ where the p_i are distinct primes, we have

$$\mu(p_1^{\alpha_1}) \cdots \mu(p_r^{\alpha_r}) = \begin{cases} (-1)^r, & \text{if } \alpha_1 = \cdots = \alpha_r = 1, \\ 0, & \text{otherwise.} \end{cases}$$

Hence $\mu(p_1^{\alpha_1}) \cdots \mu(p_r^{\alpha_r}) = \mu(n)$ as desired.

Properties of Möbius function

- We intend to prove the following identity $\mu \star \mathbf{1} = \delta$, i.e.

$$(\mu \star \mathbf{1})(n) = \sum_{d|n} \mu(d) = \delta(n) = \begin{cases} 1, & \text{if } n = 1, \\ 0, & \text{if } n > 1. \end{cases}$$

- Now since μ and $\mathbf{1}$ are multiplicative, so is the function $\mu \star \mathbf{1}$ by the previous theorem and hence $(\mu \star \mathbf{1})(1) = 1 = \delta(1)$ is true for $n = 1$.
- Besides, it is sufficient to prove $\mu \star \mathbf{1} = \delta$ for prime powers by the previous lemma. Indeed,

$$(\mu \star \mathbf{1})(p^\alpha) = \sum_{j=0}^{\alpha} \mu\left(p^j\right) = \mu(1) + \mu(p) = 1 - 1 = 0 = \delta(p^\alpha)$$

as asserted.

Möbius inversion formula

Theorem (Möbius inversion formula)

Let f and g be two arithmetic functions. Then we have

$$g = f \star \mathbf{1} \quad \Longleftrightarrow \quad f = g \star \mu$$

Equivalently, by expanding Dirichlet's convolution, we have

$$g(n) = \sum_{d|n} f(d) \iff f(n) = \sum_{d|n} g(d) \mu\left(\frac{n}{d}\right) \quad \text{for all} \quad \mathbb{Z}_+.$$

Proof: Using the identity $\mu \star \mathbf{1} = \delta$, we deduce

$$g = f \star \mathbf{1} \quad \Longleftrightarrow \quad g \star \mu = f \star (\mathbf{1} \star \mu) = f.$$

This completes the proof. □

Euler's totient function

- Euler's totient function is multiplicative and $\varphi = \mu \star \mathrm{Id}$.
- Moreover, we have

$$\varphi(p^\alpha) = p^\alpha - p^{\alpha-1} = p^\alpha \left(1 - \frac{1}{p}\right),$$

and by the multiplicativity we obtain

$$\varphi(n) = n \prod_{\substack{p|n \\ p\in\mathbb{P}}} \left(1 - \frac{1}{p}\right).$$

Theorem

If both g and $f \star g$ are multiplicative, then f is also multiplicative.

Proof: Prove it!

Further properties of multiplicative functions

Theorem

If g is multiplicative, then so is g^{-1}, its Dirichlet inverse. In particular, the set of all multiplicative functions forms a multiplicative group with multiplication defined by the Dirichlet convolution.

Proof: Prove it!

Theorem

If f is multiplicative, then we have

$$\sum_{d|n} \mu(d)f(d) = \prod_{p|n}(1 - f(p)).$$

Proof: Prove it!

von Mangoldt function

- The von Mangoldt function is an example of a function that is neither multiplicative nor additive.

Lemma

For every $n \in \mathbb{Z}_+$ we have

$$(\Lambda \star \mathbf{1})(n) = \sum_{d|n} \Lambda(d) = \log n.$$

Proof: The theorem is true if $n = 1$ since both sides are 0.

- Therefore, assume that $n > 1$ and write $n = \prod_{i=1}^{r} p_i^{\alpha_i}$. Then

$$\log n = \sum_{i=1}^{r} \alpha_i \log p_i.$$

von Mangoldt function

- The only nonzero terms in the sum $\sum_{d|n} \Lambda(d)$ come from those divisors d of the form p_k^m for $m \in [a_k]$ and $k \in [r]$. Hence, we have

$$\sum_{d|n} \Lambda(d) = \sum_{i=1}^{r} \sum_{m=1}^{a_i} \Lambda(p_i^m) = \sum_{i=1}^{r} \sum_{m=1}^{a_i} \log p_i = \sum_{i=1}^{r} a_i \log p_i = \log n.$$

- This completes the proof. □

Theorem

If $n \geq 2$, we have

$$\Lambda(n) = \sum_{d|n} \mu(d) \log \frac{n}{d} = -\sum_{d|n} \mu(d) \log d.$$

von Mangoldt function

Proof: We know that

$$(\Lambda \star \mathbf{1})(n) = \sum_{d|n} \Lambda(d) = \log n.$$

- So inverting this formula by using the Möbius inversion formula, we obtain

$$\Lambda(n) = \sum_{d|n} \mu(d) \log \frac{n}{d} = \log n \sum_{d|n} \mu(d) - \sum_{d|n} \mu(d) \log d,$$

which simplifies to

$$\Lambda(n) = \delta(n) \log n - \sum_{d|n} \mu(d) \log d.$$

- Since $\delta(n) \log n = 0$ for all $n \in \mathbb{Z}_+$, the proof is complete. □

Dirichlet series

In view of the multiplicative properties of certain arithmetic functions, we use Dirichlet series rather than power series in analytic number theory.

Definition

Let $f \in \mathbb{A}$ be an arithmetic function. The formal **Dirichlet series** of a variable $s \in \mathbb{C}$ associated to f is defined by

$$D(s, f) := \sum_{n=1}^{\infty} \frac{f(n)}{n^s}.$$

Here, we ignore convergence problems, and $D(s, f)$ is the complex number equal to the sum when it converges.

- In analytic number theory, it is customary to express a complex number $s \in \mathbb{C}$ in the form

$$s = \sigma + it \in \mathbb{C}.$$

Dirichlet series

Examples

- $D(s, \delta) = 1$.
- Presumably, the most important example of a Dirichlet series is the **Riemann zeta** function

$$D(s, \mathbf{1}) = \zeta(s) := \sum_{n=1}^{\infty} \frac{1}{n^s}.$$

Lemma

Let f, g and h be three arithmetic functions. Then

$$h = f \star g \iff D(s, h) = D(s, f) \cdot D(s, g).$$

Dirichlet series

Proof: We have

$$D(s,f)\cdot D(s,g)=\sum_{k,m=1}^{\infty}\frac{f(k)g(m)}{(km)^s}=\sum_{n=1}^{\infty}\frac{1}{n^s}\sum_{d|n}f(d)g\left(\frac{n}{d}\right)=\sum_{n=1}^{\infty}\frac{(f\star g)(n)}{n^s},$$

which completes the proof. □

Remark

The set $\mathbb{D}:=(\mathbb{D},+,\cdot)$ of formal Dirichlet series with addition $+$ and multiplication $\cdot$ defined respectively by

$$D(s,f)+D(s,g)=D(s,f+g),\quad \text{and} \quad D(s,f)\cdot D(s,g)=D(s,f\star g),$$

forms a commutative ring with additive identity 0 and multiplicative identity 1. Moreover, $\mathbb{D}:=(\mathbb{D},+,\cdot)$ is isomorphic to the ring of arithmetic functions $\mathbb{A}=(\mathbb{A},+,\star)$ via the mapping $\mathbb{A}\ni f\mapsto D(s,f)\in\mathbb{D}$.

Dirichlet series for multiplicative functions

Lemma

Let f be an arithmetic function. Then f is multiplicative if and only if

$$D(s,f) = \prod_{p \in \mathbb{P}} \left(1 + \sum_{k=1}^{\infty} \frac{f\left(p^k\right)}{p^{sk}} \right).$$

The above product is called the Euler product of $D(s,f)$.

Proof: Expanding the product we obtain a formal sum of all products of the form $\frac{f(p_1^{a_1})\cdots f(p_r^{a_r})}{(p_1^{a_1}\cdots p_r^{a_r})^s}$, where $p_1, \ldots, p_r$ are distinct prime numbers, $a_1, \ldots, a_r \in \mathbb{Z}_+$, and $r \in \mathbb{N}$.

- By multiplicativity, the numerator can be written as $f(p_1^{a_1} \cdots p_r^{a_r})$.
- The Fundamental Theorem of Arithmetic implies that the products $p_1^{a_1} \cdots p_r^{a_r}$ are in one-to-one correspondence with all natural numbers.

This gives a formal proof of the desired identity. □

Examples

- By the previous lemma $\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} = \prod_{p\in\mathbb{P}} \left(1 - \frac{1}{p^s}\right)^{-1}$, hence

$$\frac{1}{\zeta(s)} = \prod_{p\in\mathbb{P}} \left(1 - \frac{1}{p^s}\right) = \sum_{n=1}^{\infty} \frac{\mu(n)}{n^s},$$

  which implies $\mu \star \mathbf{1} = \delta$.

- Taking logarithm we obtain

$$\log \zeta(s) = \sum_{p\in\mathbb{P}} \log \left(1 - \frac{1}{p^s}\right)^{-1} = \sum_{p\in\mathbb{P}} \sum_{k=1}^{\infty} \frac{1}{kp^{ks}}.$$

- By formal differentiation, we have

$$-\zeta'(s) = \sum_{n=1}^{\infty} \frac{\log n}{n^s}, \quad \text{and} \quad -\frac{\zeta'(s)}{\zeta(s)} = \sum_{p\in\mathbb{P}} \sum_{k=1}^{\infty} \frac{\log p}{p^{ks}} = \sum_{n=1}^{\infty} \frac{\Lambda(n)}{n^s}.$$

- Thus, $\Lambda = \mu \star \log$ and consequently $\Lambda \star \mathbf{1} = \log$.

Absolute convergence of Dirichlet series

Lemma

For each Dirichlet series $D(s, f)$ with $s = \sigma + it \in \mathbb{C}$, there exists $\sigma_a \in \mathbb{R} \cup \{\pm\infty\}$, called the **abscissa of absolute convergence**, *such that*

- *$D(s, f)$ converges absolutely in the half-plane $\sigma > \sigma_a$;*
- *$D(s, f)$ does not converge absolutely in the half-plane $\sigma < \sigma_a$.*

Remarks

- In particular, the series $D(s, f)$ defines an analytic function in the halfplane $\sigma > \sigma_a$. By abuse of notation, this function will be still denoted by $D(s, f)$.
- If $|\, f(n)| \leqslant \log n$, then the series $D(s, f)$ is absolutely convergent in the half-plane $\sigma > 1$, and hence $\sigma_a \leqslant 1$.

Absolute convergence of Dirichlet series

Remarks

- At $\sigma = \sigma_a$, the series may or may not converge absolutely. For instance, $\zeta(s)$ converges absolutely in the half-plane $\sigma > \sigma_a = 1$, but does not converge on the line $\sigma = 1$.
- On the other hand, the Dirichlet series associated to the function $f(n) = 1/(\log(en))^2$ has also $\sigma_a = 1$ for the abscissa of absolute convergence, but converges absolutely at $\sigma = 1$.

Proof: Let $S := \{s \in \mathbb{C} : D(s, f) \text{ converges absolutely}\}$.

- If $S = \emptyset$, then put $\sigma_a = +\infty$. Otherwise define

$$\sigma_a := \inf\{\sigma : s = \sigma + it \in S\}.$$

- $D(s, f)$ does not converge absolutely if $\sigma < \sigma_a$ by the definition of σ_a.

Absolute convergence of Dirichlet series

- On the other hand, suppose that $D(s, f)$ is absolutely convergent for some $s_0 = \sigma_0 + it_0 \in \mathbb{C}$ and let $s = \sigma + it$ be such that $\sigma \geqslant \sigma_0$. Since

$$\left|\frac{f(n)}{n^s}\right| = \left|\frac{f(n)}{n^{s_0}}\right| \times \frac{1}{n^{\sigma-\sigma_0}} \leqslant \left|\frac{f(n)}{n^{s_0}}\right|$$

  we infer that $D(s, f)$ converges absolutely at any point s with $\sigma \geqslant \sigma_0$.

- Now by the definition of σ_a, there exist points arbitrarily close to σ_a at which $D(s, f)$ converges absolutely, and therefore by above $D(s, f)$ converges absolutely at each point s such that $\sigma > \sigma_a$. □

Simple criterium for absolute convergence

Lemma

Let $D(s,f) = \sum_{n=1}^{\infty} f(n)n^{-s}$ *be a Dirichlet series. Assume that*

$$|f(n)| \leqslant Mn^{\alpha} \quad \text{for all} \quad n \in \mathbb{Z}_+,$$

for some $\alpha \geqslant 0$ *and* $M > 0$ *independent of* n.

- *Then* $D(s,f)$ *converges absolutely in the half-plane* $\sigma > \alpha + 1$.
- *In particular,* $\sigma_a \leq \alpha + 1$.

Proof: Indeed, observe that

$$|D(s,f)| = \sum_{n=1}^{\infty} |f(n)n^{-s}| \leq M \sum_{n=1}^{\infty} n^{-(\sigma-\alpha)} < \infty,$$

whenever $\sigma > \alpha + 1$, as desired. □

Dirichlet series for products

Lemma

Let f, g be two arithmetic functions. If the Dirichlet series $D(s, f)$ and $D(s, g)$ are absolutely convergent at a point s_0, then $D(s, f \star g)$ converges absolutely at s_0 and we have $D(s_0, f \star g) = D(s_0, f)D(s_0, g)$.

Proof: We have

$$D(s_0, f)D(s_0, g) = \sum_{n=1}^{\infty} \frac{f \star g(n)}{n^{s_0}} = D(s, f \star g),$$

where the rearrangement of the terms in the double sums is justified by the absolute convergence of the two series $D(s, f)$ and $D(s, g)$ at $s = s_0$.

- Then the absolute convergence of $D(s_0, f \star g)$ follows, as we have

$$\sum_{n=1}^{\infty} \left| \frac{f \star g(n)}{n^{s_0}} \right| \leqslant \left(\sum_{n=1}^{\infty} \left| \frac{f(n)}{n^{s_0}} \right| \right) \left(\sum_{n=1}^{\infty} \left| \frac{g(n)}{n^{s_0}} \right| \right). \qquad \square$$

Dirichlet series for inverses

Corollary

Let f be an arithmetic function such that $f(1) \neq 0$. Let f^{-1} be the convolution inverse of the fuction f, i.e. $f \star f^{-1} = \delta$. Then

$$D(s, f^{-1}) = \frac{1}{D(s, f)}$$

at every point s where $D(s, f)$ and $D(s, f^{-1})$ converge absolutely.

Example

- The Möbius function satisfies $\mu^{-1} = \mathbf{1}$, hence for $\sigma > 1$, we have

$$\sum_{n=1}^{\infty} \frac{\mu(n)}{n^s} = \frac{1}{\zeta(s)}.$$

- In particular, $\sum_{n=1}^{\infty} \frac{\mu(n)}{n^2} = \frac{6}{\pi^2}$, since $\zeta(2) = \sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$.

Partial summation

Theorem

Let $f, g : \mathbb{Z}_+ \to \mathbb{C}$ *be arithmetic functions. Let* $F(x) := \sum_{1 \le n \le x} f(n)$*.*

- *Then for any* $a, b \in \mathbb{N}$ *with* $a < b$*, we have*

$$\sum_{n=a+1}^{b} f(n)g(n) = F(b)g(b) - F(a)g(a+1) - \sum_{n=a+1}^{b-1} F(n)(g(n+1) - g(n)).$$

- *Let* $x, y \in \mathbb{R}_+$ *with* $\lfloor y \rfloor < \lfloor x \rfloor$*, and let* $g \in C^1([y, x])$*. Then*

$$\sum_{y < n \le x} f(n)g(n) = F(x)g(x) - F(y)g(y) - \int_y^x F(t)g'(t)dt$$

- *In particular, if* $x \ge 2$ *and* $g \in C^1([1, x])$*, then*

$$\sum_{n \le x} f(n)g(n) = F(x)g(x) - \int_1^x F(t)g'(t)dt.$$

Quantitative estimates

Lemma

Let $D(s,f)=\sum_{n=1}^{\infty} f(n)n^{-s}$ *be a Dirichlet series. Assume that*

$$\Big|\sum_{x<n\leqslant y} f(n)\Big| \leqslant My^{\alpha} \quad \textit{for all} \quad 0<x<y,$$

for some $\alpha \geqslant 0$ *and* $M>0$ *independent of* x *and* y.

- *Then* $D(s,f)$ *converges in the half-plane* $\sigma>\alpha$.
- *Furthermore, we have in this half-plane*

$$|D(s,f)| \leqslant \frac{M|s|}{\sigma-\alpha}, \quad \textit{and} \quad \Big|\sum_{x<n\leqslant y} \frac{f(n)}{n^s}\Big| \leqslant \frac{M}{x^{\sigma-\alpha}}\left(\frac{|s|}{\sigma-\alpha}+1\right).$$

- *The latter statement ensures that* $D(s,f)$ *converges uniformly in any compact subset of the half plane* $\sigma>\alpha$.

Quantitative estimates

Proof: Set $A(x) = \sum_{1 \leq n \leqslant x} f(n)$ and $S(x,y) = A(y) - A(x)$.

- By partial summation we have

$$\sum_{x < n \leqslant y} \frac{f(n)}{n^s} = \frac{S(x,y)}{y^s} + s \int_x^y \frac{S(x,u)}{u^{s+1}} du.$$

- By hypothesis we have $|S(x,y)/y^s| \leqslant My^{\alpha - \sigma}$, so that $S(x,y)/y^s$ tends to 0 as $y \to \infty$ in the half-plane $\sigma > \alpha$.
- Therefore if one of

$$D(s,f) = \sum_{n \in \mathbb{N}} \frac{f(n)}{n^s}, \quad \text{or} \quad s \int_1^\infty \frac{A(u)}{u^{s+1}} du.$$

converges, then so does the other, and the two quantities converge to the same limit.

Quantitative estimates

- But since
$$\left|\frac{A(u)}{u^{s+1}}\right| \leqslant \frac{M}{u^{\sigma-\alpha+1}},$$
we infer that the integral converges absolutely for $\sigma > \alpha$, and hence $D(s,f)$ is convergent in this half-plane.
- Therefore for all $\sigma > \alpha$, we obtain
$$\sum_{n=1}^{\infty} \frac{f(n)}{n^s} = s\int_1^{\infty} \frac{A(u)}{u^{s+1}}\, du,$$
and hence
$$|D(s,f)| \leqslant M|s| \int_1^{\infty} \frac{du}{u^{\sigma-\alpha+1}} = \frac{M|s|}{\sigma-\alpha}.$$
- Similarly
$$\Big|\sum_{x<n\leqslant y} \frac{f(n)}{n^s}\Big| \leqslant \frac{M}{y^{\sigma-\alpha}} + M|s| \int_x^{\infty} \frac{du}{u^{\sigma-\alpha+1}} \leqslant \frac{M}{x^{\sigma-\alpha}} \left(\frac{|s|}{\sigma-\alpha} + 1\right)$$
as required. □

Conditional convergence of Dirichlet series

Lemma

For each Dirichlet series $D(s,f)$, there exists $\sigma_c \in \mathbb{R} \cup \{\pm\infty\}$, called the **abscissa of convergence**, *such that $D(s,f)$ converges in the half-plane $\sigma > \sigma_c$ and does not converge in the half-plane $\sigma < \sigma_c$. Furthermore,*

$$\sigma_c \leqslant \sigma_a \leqslant \sigma_c + 1.$$

Proof: Suppose first that $D(s,f)$ converges at a point $s_0 = \sigma_0 + it_0$ and fix a small real number $\varepsilon > 0$. By Cauchy's theorem, there exists $x_\varepsilon \geqslant 1$ such that, for all $y > x \geqslant x_\varepsilon$, we have

$$\left| \sum_{x<n\leqslant y} \frac{f(n)}{n^{s_0}} \right| \leqslant \varepsilon.$$

Conditional convergence of Dirichlet series

- Let $s = \sigma + it \in \mathbb{C}$ such that $\sigma > \sigma_0$. Using the previous lemma with s replaced by $s - s_0$ and $\alpha = 0$, we obtain

$$\Bigg| \sum_{x<n\leqslant y} \frac{f(n)}{n^s} \Bigg| \leqslant \varepsilon \left(\frac{|s-s_0|}{\sigma-\sigma_0} + 1 \right).$$

so that $D(s,f)$ converges by Cauchy's theorem.

- Now we may proceed as before. Let

$$S := \{s \in \mathbb{C} : D(s,f) \text{ converges}\}.$$

- If $S = \emptyset$, then we put $\sigma_c = +\infty$. Otherwise define

$$\sigma_c := \inf\{\sigma : s = \sigma + it \in S\}.$$

- $D(s,f)$ does not converge if $\sigma < \sigma_c$ by the definition of σ_c.

Conditional convergence of Dirichlet series

- On the other hand, there exist points s_0 with σ_0 being arbitrarily close to σ_c at which $D(s, f)$ converges.
- By above, $D(s, f)$ converges at any point s such that $\sigma > \sigma_0$. Since σ_0 may be chosen as close to σ_c as we want, it follows that $D(s, f)$ converges at any point s such that $\sigma > \sigma_c$.
- The inequality $\sigma_c \leqslant \sigma_a \leqslant \sigma_c + 1$ remains to be shown.
- The lower bound is obvious. For the upper bound, it suffices to show that if $D(s_0, f)$ converges for some s_0, then it converges absolutely for all s such that $\sigma > \sigma_0 + 1$. Now if $D(s, f)$ converges at some point s_0, then

$$\lim_{n\to\infty} f(n)n^{-s_0} = 0.$$

- Thus there exists a positive integer n_0 such that, for all $n \geqslant n_0$, we have $|f(n)| \leq n^{\sigma_0}$, hence $D(s, f)$ is absolutely convergent in the half-plane $\sigma > \sigma_0 + 1$ as required. □

Dirichlet series are holomorphic

Theorem

A Dirichlet series $D(s,f)=\sum_{n=1}^{\infty} f(n)n^{-s}$ defines a holomorphic function of the variable s in the half-plane $\sigma>\sigma_c$, in which $D(s,f)$ can be differentiated term by term so that, for all $s=\sigma+it$ with $\sigma>\sigma_c$, we have

$$\partial_s^k D(s,f)=\sum_{n=1}^{\infty}\frac{(-1)^k(\log n)^k f(n)}{n^s}, \qquad \text{for} \qquad k\in\mathbb{Z}_+.$$

Proof: The partial sums of a Dirichlet series is a holomorphic function of the variable s that converges uniformly on any compact subset of the half-plane $\sigma>\sigma_c$. Hence, $D(s,f)$ must be holomorphic in that region.

- Consequently, term-by-term differentiation is allowed, and since $n^{-s}=e^{-s\log n}$, then

$$\partial_s^k n^{-s}=(-1)^k(\log n)^k n^{-s}. \qquad \square$$

Dirichlet series are determined uniquely

Lemma

Let $D(s, f) = \sum_{n=1}^{\infty} f(n)n^{-s}$ be a Dirichlet series with abscissa of convergence σ_c.

- *If $D(s, f) = 0$ for all s such that $\sigma > \sigma_c$, then $f(n) = 0$ for all $n \in \mathbb{Z}_+$.*
- *In particular, if $D(s, f) = D(s, f)$ for all s such that $\sigma > \sigma_c$, then $f(n) = g(n)$ for all $n \in \mathbb{Z}_+$.*

Proof: Suppose the contrary and let $k \in \mathbb{Z}_+$ be the smallest integer such that $f(k) \neq 0$. Then

$$D(s, f) = \sum_{n=k}^{\infty} f(n)n^{-s} = 0$$

for all s such that $\sigma > \sigma_c$. Then we have

$$G(s) = k^s D(s, f) = k^s \sum_{n=k}^{\infty} \frac{f(n)}{n^s} = 0.$$

Dirichlet series are determined uniquely

- Therefore, for all s such that $\sigma > \sigma_c$ we have

$$G(s) = f(k) + \sum_{n=k+1}^{\infty} f(n) \left(\frac{k}{n}\right)^s = 0.$$

- Hence

$$0 = \lim_{\sigma \to \infty} G(\sigma) = f(k) \neq 0,$$

which is impossible. □

Theorem (Landau)

If $f(n) \geqslant 0$ for all $n \in \mathbb{Z}_+$, and $D(s, f) = \sum_{n=1}^{\infty} f(n) n^{-s}$ is a Dirichlet series with abscissa of convergence $\sigma_c \in \mathbb{R}$, then $D(s, f)$ has a singularity at $s = \sigma_c$.

Singularity on the axis of convergence

Proof: We may assume that $\sigma_c = 0$ and $|D(0, f)| < \infty$.

- By the Taylor expansion of $D(s, f)$ about $a > 0$ we have

$$D(s,f) = \sum_{k=0}^{\infty} \frac{(s-a)^k}{k!} \partial_s^k D(a,f) = \sum_{k=0}^{\infty} \frac{(s-a)^k}{k!} \sum_{n=1}^{\infty} \frac{(-1)^k (\log n)^k f(n)}{n^a},$$

which must converge at some $s = b < 0$. Hence

$$0 \leq \sum_{k=0}^{\infty} \sum_{n=1}^{\infty} \frac{((a-b)\log n)^k f(n)}{n^a k!} < \infty.$$

- Each term is nonnegative, so the order of summation may be changed

$$\sum_{n=1}^{\infty} \frac{f(n)}{n^a} \sum_{k=0}^{\infty} \frac{((a-b)\log n)^k}{k!} = \sum_{n=1}^{\infty} \frac{f(n)}{n^b} = \infty,$$

since $b < 0 = \sigma_c$, giving a contradiction. □

Series of multiplicative functions

Theorem

Let f be a multiplicative function satisfying

$$\sum_{p\in\mathbb{P}}\sum_{k=1}^{\infty}|f(p^k)|<\infty.$$

Then the series $\sum_{n\geqslant 1} f(n)$ is absolutely convergent and we have

$$\sum_{n=1}^{\infty} f(n)=\prod_{p\in\mathbb{P}}\left(1+\sum_{k=1}^{\infty} f\left(p^k\right)\right).$$

Proof: Let us first notice that the inequality $\sum_{p\in\mathbb{P}}\sum_{k=1}^{\infty}|f(p^k)|<\infty$ implies the convergence of the product

$$\prod_{p\in\mathbb{P}}\left(1+\sum_{k=1}^{\infty}\left|f\left(p^k\right)\right|\right)\leq\exp\left(\sum_{p\in\mathbb{P}}\sum_{k=1}^{\infty}|f(p^k)|\right)<\infty.$$

Series of multiplicative functions

- Now let $x \geqslant 2$ and set $P(x) = \prod_{p \in \mathbb{P}_{\leq x}} \left(1 + \sum_{k=1}^{\infty} \left| f(p^k) \right|\right)$.
- The convergence of the series $\sum_{k=1}^{\infty} \left| f\left(p^k\right) \right|$ enables us to rearrange the terms when we expand $P(x)$, hence
$$P(x) = \sum_{\mathrm{gpf}(n) \leq x} |f(n)|,$$
where $\mathrm{gpf}(1) = 1$ and $\mathrm{gpf}(n)$ is the greatest prime factor of $n \geq 2$.
- Since each integer $n \leqslant x$ satisfies the condition $\mathrm{gpf}(n) \leqslant x$, we have
$$\sum_{1 \leq n \leqslant x} |f(n)| \leqslant P(x)$$
- Since $P(x)$ has a finite limit as $x \to \infty$, the above inequality implies that $\sum_{n \geqslant 1} |f(n)| < \infty$. The second part of the theorem follows from
$$\left| \sum_{n=1}^{\infty} f(n) - \prod_{p \in \mathbb{P}_{\leq x}} \left(1 + \sum_{k=1}^{\infty} f(p^k)\right) \right| \leqslant \sum_{n > x} |f(n)|,$$
and the fact that the right-hand side tends to 0 as $x \to \infty$. □

Multiplication of Dirichlet series

Theorem

Let f be a multiplicative function and let $s_0 \in \mathbb{C}$. Then the three following assertions are equivalent.

(i) *One has*

$$\sum_{p \in \mathbb{P}} \sum_{k=1}^{\infty} \frac{|f(p^k)|}{p^{s_0 k}} < \infty.$$

(ii) *The series $D(s, f)$ is absolutely convergent in the half-plane $\sigma > \sigma_0$.*

(iii) *The product*

$$\prod_{p \in \mathbb{P}} \left(1 + \sum_{k=1}^{\infty} \frac{f(p^k)}{p^{sk}}\right)$$

is absolutely convergent in the half-plane $\sigma > \sigma_0$. If one of these conditions holds, then we have for all $\sigma > \sigma_0$ (in particular for all $\sigma > \sigma_a$) that

$$D(s, f) = \prod_{p \in \mathbb{P}} \left(1 + \sum_{k=1}^{\infty} \frac{f(p^k)}{p^{sk}}\right).$$