Lecture 18

Gamma function

MATH 503, FALL 2025

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Functions of finite order

- Let f be an entire function. If there exist $\rho \in \mathbb{R}_+$ and constants $A, B \in \mathbb{R}_+$ such that

$$|f(z)| \leq Ae^{B|z|^\rho} \quad \text{for all} \quad z \in \mathbb{C},$$

  then we say that f has an **order of growth** $\leq \rho$.

- We define the **order of growth** of f as

$$\rho_f = \inf \rho,$$

  where the infimum is taken over all $\rho > 0$ such that f has an order of growth $\leq \rho$.

- For example, the order of growth of the function e^{z^2} is 2.

Hadamard's theorem

Theorem

Suppose f is entire and has growth order ρ_0. Let $k \in \mathbb{Z}$ be so that $k \leq \rho_0 < k+1$. If $a_1, a_2, \ldots$ denote the (non-zero) zeros of f, then

$$f(z) = e^{P(z)} z^m \prod_{n=1}^{\infty} E_k\left(z/a_n\right),$$

where P is a polynomial of degree $\leq k$, and m is the order of the zero of f at $z=0$, and E_k are the canonical factors for $k \in \mathbb{N}$.

Example

- The function $\sin \pi z$ is entire and of order one, and its zeros are at $z = 0, \pm 1, \pm 2, \ldots$, and so, by Hadamard's theorem we can write

$$\sin \pi z = z e^{H(z)} \prod_{n=1}^{\infty} \left(1 - \frac{z^2}{n^2}\right),$$

where $H(z) = az + b$.

- Taking the logarithmic derivative of this equation, we find that

$$\pi \frac{\cos \pi z}{\sin \pi z} = \frac{1}{z} + H'(z) - \sum_{n=1}^{\infty} \frac{2z}{n^2 - z^2}.$$

- Passage to the limit as $z \to 0$ gives $a = 0$, and so $H(z) = b$. Thus,

$$\frac{\sin \pi z}{z} = c \prod_{n=1}^{\infty} \left(1 - \frac{z^2}{n^2}\right).$$

Example

- Passing again to the limit as $z \to 0$ gives $c = \pi$, i.e.

$$\sin \pi z = \pi z \prod_{n=1}^{\infty} \left(1 - \frac{z^2}{n^2}\right).$$

- Equivalently, we have

$$\frac{\sin \pi z}{\pi z} = \prod_{n=1}^{\infty} \left(1 - \frac{z^2}{n^2}\right).$$

Euler's gamma function

- The Euler gamma function $\Gamma(z)$ is defined by the equation

$$\frac{1}{\Gamma(z)} = ze^{\gamma z} \prod_{n=1}^{\infty} \left(1 + \frac{z}{n}\right) e^{-z/n}$$

  where γ is Euler's constant.
- It follows from the definition that $\Gamma^{-1}(z)$ is an entire function of order one. Prove it! In, fact one can show that there are $A, B \in \mathbb{R}_+$ so that

$$\frac{1}{|\Gamma(z)|} \leq Ae^{B|z| \log |z|}.$$

- Moreover, $\Gamma(z)$ is an analytic function in the entire $\mathbb{C}$ except for the points $s = 0, -1, -2, \ldots$, where it has simple poles.

Euler's gamma function

Theorem (Euler's formula)

For every $z \in \mathbb{C} \setminus \{-n : n \in \mathbb{N}\}$, we have

$$\Gamma(z) = \frac{1}{z} \prod_{n=1}^{\infty} \left(1 + \frac{1}{n}\right)^{z} \left(1 + \frac{z}{n}\right)^{-1}.$$

In other words, $\Gamma(z)$ is a meromorphic function on $\mathbb{C}$ with simple poles at 0 and at the negative integers and with no zeros.

Proof

- From the definition of an infinite product and from the definition of the function $\Gamma(z)$, we obtain

$$\begin{aligned}
\frac{1}{\Gamma(z)} &= z \lim_{m\to\infty} e^{z\left(1+\frac{1}{2}+\ldots+\frac{1}{m}-\log m\right)} \cdot \lim_{m\to\infty} \prod_{n=1}^{m} \left(1+\frac{z}{n}\right) e^{-\frac{z}{n}} \\
&= z \lim_{m\to\infty} m^{-z} \prod_{n=1}^{m} \left(1+\frac{z}{n}\right) \\
&= z \lim_{m\to\infty} \prod_{n=1}^{m-1} \left(1+\frac{1}{n}\right)^{-z} \prod_{n=1}^{m} \left(1+\frac{z}{n}\right) \\
&= z \lim_{m\to\infty} \prod_{n=1}^{m} \left(1+\frac{1}{n}\right)^{-z} \left(1+\frac{z}{n}\right) \left(1+\frac{1}{m}\right)^{z} \\
&= z \prod_{n=1}^{\infty} \left(1+\frac{1}{n}\right)^{-z} \left(1+\frac{z}{n}\right). \qquad \square
\end{aligned}$$

Properties of Gamma function

Corollary

For every $z \in \mathbb{C} \setminus \{-n : n \in \mathbb{N}\}$, *we have*

$$\Gamma(z) = \lim_{n\to\infty} \frac{(n-1)! \cdot n^z}{z(z+1)\cdot \ldots \cdot (z+n-1)}.$$

Proof: From the previous theorem we have

$$\Gamma(z) = \lim_{n\to\infty} z^{-1} \prod_{m=1}^{n-1} \left(1+\frac{1}{m}\right)^z \left(1+\frac{z}{m}\right)^{-1}$$

$$= \lim_{n\to\infty} \frac{2^z \cdot \frac{3^z}{2^z} \cdot \ldots \cdot \frac{n^z}{(n-1)^z}}{z \cdot \frac{(z+1)}{1} \cdot \ldots \cdot \frac{(z+n-1)}{n-1}} = \lim_{n\to\infty} \frac{1\cdot 2\cdot \ldots \cdot (n-1)n^z}{z\cdot (z+1)\cdot \ldots \cdot (z+n-1)}. \quad \square$$

Corollary

We also have $\Gamma(1) = \Gamma(2) = 1$.

Properties of Gamma function

Theorem (Functional equation)

- *We have* $\Gamma(z+1) = z\Gamma(z)$ *for all* $z \in \mathbb{C} \setminus \{-n : n \in \mathbb{N}\}$.
- *In particular,* $\Gamma(n+1) = n!$ *for all* $n \in \mathbb{N}$, *and* $\operatorname{res}_{z=-m}\Gamma(z) = \frac{(-1)^m}{m!}$.

Proof: We have

$$
\begin{aligned}
\frac{\Gamma(z+1)}{\Gamma(z)} &= \frac{z}{z+1} \lim_{m\to\infty} \prod_{n=1}^{m} \frac{\left(1+\frac{1}{n}\right)^{z+1}\left(1+\frac{z+1}{n}\right)^{-1}}{\left(1+\frac{1}{n}\right)^{z}\left(1+\frac{z}{n}\right)^{-1}} \\
&= \frac{z}{z+1} \lim_{m\to\infty} \prod_{n=1}^{m} \frac{n+1}{n} \cdot \frac{n+z}{n+z+1} \\
&= \frac{z}{z+1} \lim_{m\to\infty} \frac{(m+1)(z+1)}{m+1+z} = z.
\end{aligned}
$$

This completes the proof. □

Properties of Gamma function

Corollary (Duplication formula)

$$\Gamma(2z)\Gamma(1/2) = 2^{2z-1}\Gamma(z)\Gamma(z+1/2) \quad \textit{for all} \quad z \in \mathbb{C} \setminus (-\mathbb{N}).$$

Theorem (Reflection formula)

$$\frac{\sin \pi z}{\pi} = \frac{1}{\Gamma(z)\Gamma(1-z)} \quad \textit{for all} \quad z \in \mathbb{C}.$$

Proof: We know that $\frac{\sin \pi z}{\pi z} = \prod_{n=1}^{\infty}\left(1 - \frac{z^2}{n^2}\right)$. On the other hand,

$$\frac{1}{\Gamma(z)\Gamma(-z)} = -z^2 \prod_{n=1}^{\infty}\left(1 - \frac{z^2}{n^2}\right)$$

But we also know that $\Gamma(1-z) = -z\Gamma(-z)$, and the result follows.

Corollary

As a corollary we obtain that $\Gamma(1/2) = \sqrt{\pi}$.

Integral representation of the gamma function

Theorem (Integral representation)

Suppose that $\mathrm{Re}(z) > 0$. *Then*

$$\Gamma(s) = \int_0^\infty e^{-t} t^{z-1} dt.$$

Proof: We know that

$$\Gamma(s) = \lim_{n\to\infty} \frac{n! \cdot n^z}{z(z+1)(z+2)\cdots(z+n)}.$$

- We have to establish two things. Firstly, we will show that

$$\int_0^n \left(1 - \frac{t}{n}\right)^n t^{z-1} dt = \frac{n! \cdot n^z}{z(z+1)\cdot \ldots \cdot (z+n)} \quad \text{for all} \quad n \in \mathbb{Z}_+.$$

Integral representation of the gamma function

- Secondly, we will show that

$$\lim_{n\to\infty}\int_0^n \left(1-\frac{t}{n}\right)^n t^{z-1}dt = \int_0^\infty e^{-t}t^{z-1}dt.$$

- Indeed, when $s>0$ the above integral converges and we have

$$\begin{aligned}\int_0^n \left(1-\frac{t}{n}\right)^n t^{z-1}dt &= n^z\int_0^1 (1-u)^n u^{z-1}du = n^z\frac{n}{z}\int_0^1 (1-u)^{n-1}u^z du\\ &= n^z\frac{n(n-1)}{z(z+1)}\int_0^1 (1-u)^{n-2}u^{z+1}du\\ &\vdots\\ &= n^s\frac{n(n-1)\cdot\ldots\cdot 1}{z(z+1)\cdot\ldots\cdot(z+n-1)}\int_0^1 u^{z+n-1}du\\ &= \frac{n!\cdot n^z}{z(z+1)(z+2)\cdot\ldots\cdot(z+n)}.\end{aligned}$$

Integral representation of the gamma function

- Thus, it suffices to prove that

$$\lim_{n\to\infty}\int_0^n \left(1-\frac{t}{n}\right)^n t^{z-1}dt = \int_0^\infty e^{-t}t^{z-1}dt.$$

- To this end, we consider the functions

$$f_n(t) = \begin{cases} (1-t/n)^n t^{z-1} & \text{if } 0 \leq t \leq n, \\ 0 & \text{if } t > n. \end{cases}$$

- Each of these functions is in $L^1([0,\infty))$ and satisfies the inequality

$$|f_n(t)| \leq e^{-t}t^{\sigma-1}, \qquad \text{where} \qquad \sigma = \mathrm{Re}(z).$$

- The last inequality is easily verified by taking logarithms and noting

$$n\log\left(1-\frac{t}{n}\right) = -t - \frac{t^2}{2n} - \frac{t^3}{3n^2} - \cdots < -t.$$

Integral representation of the gamma function

- Furthermore,

$$\lim_{n\to\infty} f_n(t) = t^{s-1} \lim_{n\to\infty} \left(1 - \frac{t}{n}\right)^n = e^{-t} t^{s-1}.$$

- Since the function $e^{-t} t^{\sigma-1}$ is in $L^1([0,\infty))$, the dominated convergence theorem yields

$$\lim_{n\to\infty} \int_0^\infty f_n(t) dt = \int_0^\infty \lim_{n\to\infty} f_n(t) dt = \int_0^\infty e^{-t} t^{s-1} dt,$$

which completes the proof of the lemma. □

Bernoulli's numbers and polynomials

- For a complex number $x \in \mathbb{C}$, we observe that

$$u = u(x,z) := \frac{ze^{xz}}{e^z - 1}$$

is analytic in $|z| < 2\pi$.

- Therefore, it has power series expansion around $z = 0$ given by

$$u = u(x,z) = \sum_{k=0}^{\infty} \frac{B_k(x)}{k!} z^k \quad \text{in} \quad |z| < 2\pi,$$

where $B_k(x)$ are polynomials in the variable $x \in \mathbb{C}$ with rational coefficients, known as the **Bernoulli polynomials**.

- Further $B_k = B_k(0) \in \mathbb{Q}$ are called the **Bernoulli numbers** given by

$$\frac{z}{e^z - 1} = \sum_{k=0}^{\infty} \frac{B_k}{k!} z^k$$

derived from the above with $x = 0$.

Bernoulli's numbers and polynomials

- Differentiating the power series k times with respect to z, we obtain

$$\left.\frac{\partial^k u}{\partial z^k}\right|_{z=0} = B_k(x) \quad \text{for} \quad k \geq 0.$$

- We have

$$\frac{z}{e^z - 1} = \frac{z}{\sum_{k=1}^{\infty} \frac{z^k}{k!}} = \left(\sum_{k=0}^{\infty} \frac{z^k}{(k+1)!}\right)^{-1}.$$

- Then we see from these expansions that $B_0 = 1$, $B_1 = -\frac{1}{2}$, and

$$\left(\sum_{k=0}^{\infty} \frac{z^k}{(k+1)!}\right)\left(\sum_{k=0}^{\infty} \frac{B_k}{k!} z^k\right) = 1.$$

- The left-hand side is equal to $\sum_{m=1}^{\infty} c_{m-1} z^{m-1}$, where

$$c_{m-1} = \sum_{k=0}^{m-1} \frac{B_k}{k!} \frac{1}{(m-k)!}.$$

Bernoulli's numbers and polynomials

- Hence, we obtain the following reccurence

$$\frac{B_0}{0!m!} + \frac{B_1}{1!(m-1)!} + \cdots + \frac{B_{m-1}}{(m-1)!1!} = \begin{cases} 1 & \text{if } m = 1, \\ 0 & \text{if } m > 1. \end{cases}$$

- Further, we see that

$$\frac{z}{e^z - 1} + \frac{z}{2} = 1 + \sum_{k=2}^{\infty} \frac{B_k}{k!} z^k.$$

- Since the left-hand side is an even function of z, we derive that

$$B_k = 0 \quad \text{for} \quad k = 2m+1 \text{ and } m \in \mathbb{N}.$$

- Further, we compute B_k for $2 \leq k \leq 14$ as follows: $B_2 = \frac{1}{6}$, and

$$B_4 = -\frac{1}{30}, B_6 = \frac{1}{42}, B_8 = -\frac{1}{30}, B_{10} = \frac{5}{66}, B_{12} = -\frac{691}{2730}, B_{14} = \frac{7}{6}.$$

Bernoulli's numbers and polynomials

Lemma

The Bernoulli polynomials $B_k(x)$ with $k \geq 0$ satisfy the following:

(a) *$B_k(x)$ is a monic polynomial of degree k given by*

$$B_k(x) = \sum_{m=0}^{k} \binom{k}{m} B_m x^{k-m}.$$

(b) *We have*

$$B_k(1) - B_k(0) = \begin{cases} 1 & \text{if } k = 1 \\ 0 & \text{if } k \neq 1. \end{cases}$$

(c) *Also, $B_k(1-x) = (-1)^k B_k(x)$.*

(d) *Finally, $B_k'(x) = kB_{k-1}(x)$ for $k > 0$.*

Bernoulli's numbers and polynomials

Proof of (a): We have

$$\sum_{k=0}^{\infty} \frac{B_k(x)}{k!} z^k = \frac{z}{e^z - 1} e^{xz} = \left(\sum_{k=0}^{\infty} \frac{B_k}{k!} z^k \right) \left(\sum_{k=0}^{\infty} \frac{x^k}{k!} z^k \right).$$

- Now the assertion follows immediately by comparing the coefficients of z^k on both sides. Further, the coefficient of x^k in $B_k(x)$ is $\binom{k}{0} B_0 = 1$. Hence, $B_k(x)$ is a monic polynomial of degree k. □

Proof of (b): Note that

$$u(1, z) - u(0, z) = \frac{z}{e^z - 1} e^z - \frac{z}{e^z - 1} = z.$$

- By differentiating both sides k times, we see that $B_k(1) - B_k(0) = 1$ if $k = 1$ and 0 otherwise. □

Bernoulli's numbers and polynomials

Proof of (c): Note that

$$u(1-x,z) = \frac{z}{e^z - 1} e^{(1-x)z} = \frac{-z}{e^{-z} - 1} e^{-xz} = u(x,-z).$$

- By differentiating both sides k times, we derive that $B_k(1-x) = (-1)^k B_k(x)$. □

Proof of (d): We have $\frac{\partial}{\partial x} u(x,z) = zu(x,z)$, which we rewrite as

$$\sum_{k=1}^{\infty} \frac{B_k'(x)}{k!} z^k = \sum_{k=1}^{\infty} \frac{B_{k-1}(x)}{(k-1)!} z^k.$$

- By comparing the coefficients of z^k on both sides, we obtain

$$\frac{B_k'(x)}{k!} = \frac{B_{k-1}(x)}{(k-1)!} \quad \text{for} \quad k > 0,$$

which implies the assertion. □

Euler–Maclaurin–Jacobi summation formula

Theorem

Let $b > a$ and $q \geq 1$ be integers. Let $f \in C^q([a, b])$, then

$$\sum_{n=a+1}^{b} f(n) = \int_a^b f(x)dx + \sum_{r=1}^{q} (-1)^r \frac{B_r}{r!} \left(f^{(r-1)}(b) - f^{(r-1)}(a) \right) + R_q,$$

where

$$R_q = \frac{(-1)^{q+1}}{q!} \int_a^b B_q(x - \lfloor x \rfloor) f^{(q)}(x)dx.$$

Proof: Let F be q times continuously differentiable in $[0, 1]$.

- By the previous lemma (a) and (d) with $k = 1$, we have

$$B_1(x) = B_0 x + B_1 = x - \frac{1}{2}, \quad \text{and} \quad B_1'(x) = 1.$$

Euler–Maclaurin–Jacobi summation formula

- Then
$$\int_0^1 F(x)dx = \int_0^1 F(x)B_1'(x)dx.$$
- Integrating the right-hand side by parts, we derive from the previous lemma (d) with $k = 2$ that
$$\int_0^1 F(x)B_1'(x)dx = \frac{F(1)+F(0)}{2} - \int_0^1 F'(x)B_1(x)dx$$
$$= \frac{F(1)+F(0)}{2} - \frac{1}{2}\int_0^1 F'(x)B_2'(x)dx.$$
- By the previous lemma (b) with $k > 1$ and (d) with $k = 3$, we obtain
$$\int_0^1 F'(x)B_2'(x)dx = B_2\left(F'(1) - F'(0)\right) - \int_0^1 F''(x)B_2(x)dx$$
$$= B_2\left(F'(1) - F'(0)\right) - \frac{1}{3}\int_0^1 F''(x)B_3'(x)dx.$$

Euler–Maclaurin–Jacobi summation formula

- Now we proceed inductively, as above, for obtaining

$$\int_0^1 F(x)dx = \frac{F(1)+F(0)}{2} + \sum_{r=2}^{q}(-1)^{r-1}\frac{B_r}{r!}\left(F^{(r-1)}(1) - F^{(r-1)}(0)\right)$$
$$+\frac{(-1)^q}{q!}\int_0^1 B_q(x)F^{(q)}(x)dx.$$

- Since $B_1 = -\frac{1}{2}$, we obtain

$$\frac{F(1)+F(0)}{2} = F(1) - \frac{F(1)-F(0)}{2} = F(1) + B_1(F(1)-F(0)).$$

- Consequently, we have

$$F(1) = \int_0^1 F(x)dx + \sum_{r=1}^{q}(-1)^{r}\frac{B_r}{r!}\left(F^{(r-1)}(1) - F^{(r-1)}(0)\right)$$
$$+\frac{(-1)^{q+1}}{q!}\int_0^1 B_q(x)F^{(q)}(x)dx.$$

Euler–Maclaurin–Jacobi summation formula

- For positive integer n with $a \le n \le b$, let $F(x) = f(n-1+x)$.
- Then $F(x)$ is q times continuously differentiable in $[0,1]$ since f is q times continuously differentiable in $[a,b]$.
- Then we derive from the above formula that

$$f(n) = \int_{n-1}^{n} f(x)dx + \sum_{r=1}^{q} (-1)^r \frac{B_r}{r!} \left(f^{(r-1)}(n) - f^{(r-1)}(n-1) \right) + \frac{(-1)^{q+1}}{q!} \int_0^1 B_q(x) f^{(q)}(n-1+x)dx.$$

- Letting n run from $a+1$ to b, we obtain

$$\sum_{n=a+1}^{b} f(n) = \int_a^b f(x)dx + \sum_{r=1}^{q} (-1)^r \frac{B_r}{r!} \left(f^{(r-1)}(b) - f^{(r-1)}(a) \right) + R_q,$$

where

$$R_q = \frac{(-1)^{q+1}}{q!} \sum_{n=a+1}^{b} \int_0^1 B_q(x) f^{(q)}(n-1+x)dx.$$

Euler–Maclaurin–Jacobi summation formula

- For $n = a + r$ with $1 \leq r \leq b - a$, we have

$$\int_0^1 B_q(x) f^{(q)}(n - 1 + x) dx = \int_0^1 B_q(x) f^{(q)}(a + r - 1 + x) dx$$

- Putting $a + r - 1 + x = y$, the above integral is equal to

$$\int_{a+r-1}^{a+r} B_q(y - \lfloor y \rfloor) f^{(q)}(y) dy.$$

- Hence

$$R_q = \frac{(-1)^{q+1}}{q!} \int_a^b B_q(x - \lfloor x \rfloor) f^{(q)}(x) dx.$$

- This completes the Euler–Maclaurin–Jacobi summation formula. □

Stirling formula

Theorem

Let $m \in \mathbb{N}$. For all $z \in \mathbb{C} \setminus \{-n \in \mathbb{Z} : n \geq 0\}$, we have

$$\log \Gamma(z) = \left(z - \frac{1}{2}\right) \log z - z + \frac{1}{2} \log 2\pi + K_m(z), \qquad (*)$$

where logarithm has principal value and

$$K_m(z) = \sum_{j=1}^{m} \frac{B_{2j}}{(2j-1)2j} \frac{1}{z^{2j-1}} - \frac{1}{2m} \int_0^\infty \frac{B_{2m}(x - \lfloor x \rfloor)}{(x+z)^{2m}} dx.$$

Proof: We check that both the sides in are holomorphic functions of z in the region $\mathbb{C} \setminus (-\infty, 0]$. Therefore, by the identity theorem, it suffices to prove (*) for all real numbers $z \geq z_0$ where $z_0 > 0$ is sufficiently large.

Stirling formula

- By the properties of the Γ function, we have

$$\Gamma(z) = (z-1)\Gamma(z-1) = \prod_{n=1}^{\infty}\left(\left(1+\frac{z-1}{n}\right)^{-1}\left(1+\frac{1}{n}\right)^{z-1}\right)$$
$$= \lim_{N\to\infty}\left((N+1)^{z-1}\prod_{n=1}^{N}\left(1+\frac{z-1}{n}\right)^{-1}\right).$$

- Since $\Gamma(z)$ has no zero and it has pole at zero and at negative integer and none of the term in the above product vanishes, we derive that

$$\log\Gamma(z) = \lim_{N\to\infty}\left((z-1)\log N - \sum_{n=1}^{N}\log\left(\frac{n+z-1}{n}\right)\right).$$

- Now we apply the Euler–Maclaurin–Jacobi formula to the sum above with $a=1, b=N$, $f(x)=\log(x+z-1)-\log x$ and $q=2m$.

Stirling formula

- For $x \in [a, b]$ and $1 \leq r \leq 2m$, we observe that

$$f^{(r)}(x) = (-1)^{r-1}(r-1)! \left(\frac{1}{(x+z-1)^r} - \frac{1}{x^r} \right).$$

- Hence we conclude

$$\sum_{n=1}^{N} \log\left(\frac{n+z-1}{n}\right) = \log z + \sum_{n=2}^{N} \log\left(\frac{n+z-1}{n}\right) = \log z$$
$$+ \int_1^N (\log(x+z-1) - \log x)dx + \frac{1}{2}(\log(N+z-1) - \log N - \log z)$$
$$+ \sum_{j=1}^{m} \frac{B_{2j}}{(2j-1)2j} \left(\frac{1}{(N+z-1)^{2j-1}} - \frac{1}{N^{2j-1}} - \frac{1}{z^{2j-1}} + 1 \right)$$
$$+ \frac{1}{2m} \int_1^N B_{2m}(x - \lfloor x \rfloor) \left(\frac{1}{(x+z-1)^{2m}} - \frac{1}{x^{2m}} \right) dx, \qquad (**)$$

since $B_1 = -\frac{1}{2}$ and $B_3 = B_5 = \cdots = B_{2m-1} = 0$.

Stirling formula

- Integrating by parts the first integral in (**). we obtain

$$\int_1^N \log(x+z-1)dx = (N+z-1)\log(N+z-1) - z\log z - N + 1,$$

and

$$\int_1^N \log x dx = N\log N - N + 1.$$

- Further

$$\lim_{N\to\infty} \frac{1}{2}(\log(N+z-1) - \log N - \log z) = -\frac{1}{2}\log z.$$

- Also note that

$$\int_1^\infty \frac{B_{2m}(x-\lfloor x\rfloor)}{(x+z-1)^{2m}}dx = \int_0^\infty \frac{B_{2m}(x-\lfloor x\rfloor)}{(x+z)^{2m}}dx$$

Stirling formula

- Hence, we obtain

$$\lim_{N\to\infty}\left(\sum_{j=1}^{m}\frac{B_{2j}}{(2j-1)2j}\left(\frac{1}{(N+z-1)^{2j-1}}-\frac{1}{N^{2j-1}}-\frac{1}{z^{2j-1}}+1\right)\right.$$
$$\left.+\frac{1}{2m}\int_1^N B_{2m}(x-\lfloor x\rfloor)\left(\frac{1}{(x+z-1)^{2m}}-\frac{1}{x^{2m}}\right)dx\right)$$
$$=-K_m(z)-L'_m,$$

where

$$K_m(z)=\sum_{j=1}^{m}\frac{B_{2j}}{(2j-1)2j}\frac{1}{z^{2j-1}}-\frac{1}{2m}\int_0^\infty\frac{B_{2m}(x-\lfloor x\rfloor)}{(x+z)^{2m}}dx,$$

and

$$L'_m=\frac{1}{2m}\int_1^\infty\frac{B_{2m}(x-\lfloor x\rfloor)}{x^{2m}}dx-\sum_{j=1}^{m}\frac{B_{2j}}{(2j-1)2j}.$$

Stirling formula

- Therefore, we conclude that

$$\log \Gamma(z) = A + \lim_{n\to\infty} B(N),$$

where

$$A = \left(z - \frac{1}{2}\right) \log z + K_m(z) + L'_m,$$

and

$$\begin{aligned} B(N) &= -(N+z-1)\log(N+z-1) + (N+z-1)\log N \\ &= -(N+z-1)\log\left(1 + \frac{z-1}{N}\right), \end{aligned}$$

satisfying

$$\lim_{N\to\infty} B(N) = -z + 1.$$

Stirling formula

- Hence, we have proved that

$$\log\Gamma(z) = \left(z - \frac{1}{2}\right)\log z - z + K_m(z) + L_m,$$

where $L_m = L'_m + 1$. Since $\lim_{z\to\infty} K_m(z) = 0$, we have

$$\lim_{z\to\infty}\left(\log\Gamma(z) - \left(z - \frac{1}{2}\right)\log z + z\right) = L_m.$$

- The proof will be competed if we prove that $L_m = \frac{1}{2}\log 2\pi$.
- Taking $z = N + 1$ and using $\Gamma(N+1) = N!$, we have

$$\begin{aligned} L_m &= \lim_{N\to\infty}\left(\log N! - \left(N + \frac{1}{2}\right)\log(N+1) + N + 1\right) \\ &= \lim_{N\to\infty}\left(\log N! - \left(N + \frac{1}{2}\right)\log N + N\right), \end{aligned}$$

since $\log(N+1) = \log N + O(1/N)$.

Stirling formula

- Therefore
$$\lim_{N\to\infty} \frac{N!}{N^N N^{1/2} e^{-N}} = e^{Lm}.$$
- Setting $z = \frac{1}{2}$ in both sides of $\frac{\sin \pi z}{\pi z} = \prod_{n=1}^{\infty}\left(1 - \frac{z^2}{n^2}\right)$, we obtain
$$\prod_{n=1}^{\infty}\left(1 - \frac{1}{4n^2}\right)^{-1} = \frac{\pi}{2}.$$
- Thus
$$\frac{\pi}{2} = \lim_{N\to\infty} \prod_{n=1}^{N} \frac{4n^2}{(2n-1)(2n+1)} = \lim_{N\to\infty} \frac{4^{2N}(N!)^4}{((2N)!)^2(2N+1)}$$
by rewriting
$$\prod_{n=1}^{N}(2n-1)(2n+1) = \frac{((2N)!)^2(2N+1)}{(2\cdot 4\cdots 2N)^2} = \frac{((2N)!)^2(2N+1)}{4^N(N!)^2}.$$

Stirling formula

- Now, we derive that

$$\frac{\pi}{2} = \lim_{N\to\infty} \frac{4^{2N} N^{4N} e^{-4N} N^2 e^{4L_m}}{(2N)^{4N} e^{-4N} 2N(2N+1) e^{2L_m}} = \frac{1}{4} e^{2L_m},$$

which implies that $L_m = \frac{1}{2}\log(2\pi)$ as desired. □

Corollary

Let $0 < \delta < \pi$, then for any $z \in \mathbb{C}$ so that $|\arg z| < \pi - \delta$, we have

$$\log \Gamma(z) = \left(z - \frac{1}{2}\right)\log z - z + \frac{1}{2}\log 2\pi + O(|z|^{-1}), \qquad (**)$$

uniformly as $|z| \to \infty$, where logarithm has principal value, and the implicit constant depend at most on δ.

Stirling formula

Proof: If we apply the previous theorem with $m = 1$, we obtain

$$\log \Gamma(z) = \left(z - \frac{1}{2}\right) \log z - z + \frac{1}{2} \log 2\pi + K_1(z),$$

where

$$K_1(z) = \frac{1}{12z} - \frac{1}{2} \int_0^\infty \frac{B_2(x - \lfloor x \rfloor)}{(x+z)^2}\, dx = -\frac{1}{2} \int_0^\infty \frac{B_2(x - \lfloor x \rfloor) - B_2}{(x+z)^2}\, dx,$$

and $B_2(x) - B_2 = x^2 - x$.

- Note that

$$\left| \int_0^\infty \frac{B_2(x - \lfloor x \rfloor) - B_2}{(x+z)^2}\, dx \right| \leq \frac{1}{4} \int_0^\infty \frac{dx}{|x+z|^2},$$

since $B_2(x - \lfloor x \rfloor) - B_2 \leq 1/4$.

Stirling formula

- If $\theta = \arg z$, then $z = re^{i\theta}$ and we can write

$$\begin{aligned}\int_0^\infty \frac{dx}{|x+z|^2} &= \int_0^\infty \frac{dx}{x^2 + r^2 + 2xr\cos\theta} \\ &\leq \int_0^\infty \frac{dx}{x^2 + r^2 - 2xr\cos\delta},\end{aligned}$$

  since $|\theta| = |\arg z| < \pi - \delta$.
- Since $2xr \leq x^2 + r^2$, then

$$\int_0^\infty \frac{dx}{x^2 + r^2 - 2xr\cos\delta} \leq \frac{1}{1-\cos\delta}\int_0^\infty \frac{dx}{x^2+r^2} \leq \frac{\pi}{2(1-\cos\delta)r}.$$

- In fact we proved that

$$|K_1(z)| \leq \frac{\pi}{16(1-\cos\delta)r}.$$

- This completes the proof. □

Stirling formula

Corollary

Let $0 < \delta < \pi$, *then for any* $z \in \mathbb{C}$ *so that* $|\arg z| < \pi - \delta$, *we have*

$$\frac{\Gamma'(z)}{\Gamma(z)} = \log z - \frac{1}{2z} + O(|z|^{-2}), \tag{***}$$

uniformly as $|z| \to \infty$, *where logarithm has principal value, and the implicit constant depend at most on* δ.

Proof: It suffices to differentiate the formula

$$\log\Gamma(z) = \left(z - \frac{1}{2}\right)\log z - z + \frac{1}{2}\log 2\pi + K_1(z),$$

where

$$K_1(z) = -\frac{1}{2}\int_0^\infty \frac{B_2(x - \lfloor x \rfloor) - B_2}{(x+z)^2}\,dx, \qquad \text{and} \qquad B_2(x) - B_2 = x^2 - x.$$

Stirling formula

- Then we have
$$\frac{\Gamma'(z)}{\Gamma(z)} = \log z - \frac{1}{2z} + K_1'(z),$$
where
$$K_1'(z) = \int_0^\infty \frac{B_2(x - \lfloor x \rfloor) - B_2}{(x+z)^3}\, dx,$$
- Observe that
$$|K_1'(z)| \leq \frac{1}{4} \int_0^\infty \frac{dx}{|x+z|^3}.$$
- If $\theta = \arg z$, then $z = re^{i\theta}$ and $|\theta| = |\arg z| < \pi - \delta$, and we can write
$$\begin{aligned}\int_0^\infty \frac{dx}{|x+z|^3} &= \int_0^\infty \frac{dx}{(x^2 + r^2 + 2xr\cos\theta)^{3/2}} \\ &\leq \int_0^\infty \frac{dx}{(x^2 + r^2 - 2xr\cos\delta)^{3/2}}.\end{aligned}$$

Stirling formula

- Since $2xr \leq x^2 + r^2$, then

$$\int_0^\infty \frac{dx}{(x^2+r^2-2xr\cos\delta)^{3/2}} \leq \frac{1}{(1-\cos\delta)^{3/2}} \int_0^\infty \frac{dx}{(x^2+r^2)^{3/2}}$$
$$\leq \frac{\pi}{2(1-\cos\delta)^{3/2} r^2}.$$

- In fact we proved that

$$|K_1'(z)| \leq \frac{\pi}{8(1-\cos\delta)^{3/2} r^2}.$$

- This completes the proof. □

Stirling formula

Lemma

Let $z = \sigma + it$ with $z \neq 0$ such that either $\sigma > 0, t = 0$ or $t \neq 0$. Then

$$|K_1(z)| \leq \begin{cases} \frac{1}{8\sigma} & \text{if } \sigma > 0, t = 0, \\ \frac{1}{8|t|} \arctan \frac{|t|}{\sigma} & \text{if } t \neq 0, \end{cases}$$

where

$$0 \leq \arctan \frac{|t|}{\sigma} = |\arg z| < \pi.$$

Proof: By the previous theorem with $m = 1$, we have

$$K_1(z) = -\frac{1}{2} \int_0^{\infty} \frac{B_2(x - \lfloor x \rfloor)}{(x+z)^2} dx + \frac{B_2}{2z} = -\frac{1}{2} \int_0^{\infty} \frac{B_2(x - \lfloor x \rfloor) - B_2}{(x+z)^2} dx.$$

Stirling formula

- By the properties of Bernoulli's polynomials $B_2(x) - B_2 = x^2 - x$.
- Therefore
$$|B_2(x - \lfloor x \rfloor) - B_2| \leq \frac{1}{4}.$$
and hence
$$|K_1(z)| \leq \frac{1}{8}\int_0^\infty \frac{dx}{(\sigma + x)^2 + t^2}.$$
- We may assume that $t > 0$. Let $\sigma > 0$. By putting $\sigma + x = t\tan\theta$, the integral is equal to
$$\frac{1}{t}\int_{\frac{\pi}{2} - \arctan\frac{t}{\sigma}}^{\frac{\pi}{2}} d\theta = \frac{1}{t}\arctan\frac{t}{\sigma},$$
and the assertion follows. Here, we have used the identity
$$\arctan(u) = \frac{\pi}{2} - \arctan\left(\frac{1}{u}\right)$$
for $0 < u \leq 1$. The proof for the case $\sigma \leq 0$ is similar. □

Stirling formula

Corollary

Let $a, b \in \mathbb{R}$ be fixed and $a \leq b$.

(i) *Then for every $z = \sigma + it$ with $\sigma \in [a, b]$ and $|t| \geq 1$, we have*

$$\Gamma(z) = \sqrt{2\pi} e^{-\frac{\pi}{2}|t|} |t|^{\sigma - \frac{1}{2}} e^{i|t|(\log|t| - 1)} e^{\frac{\pi i}{2}(\sigma - \frac{1}{2})} \left(1 + O\left(\frac{1}{|t|}\right)\right).$$

(ii) *Moreover,* $|\Gamma(z)| = \sqrt{2\pi} e^{-\frac{\pi}{2}|t|} |t|^{\sigma - \frac{1}{2}} \left(1 + O\left(\frac{1}{|t|}\right)\right)$.

(iii) *This implies* $|\Gamma(z)| = O\left(e^{-\frac{\pi}{2}|t|} |t|^{\sigma - \frac{1}{2}}\right)$ *and*

(iv) $\frac{1}{|\Gamma(z)|} = O\left(e^{\frac{\pi}{2}|t|} |t|^{\frac{1}{2} - \sigma}\right)$.

Stirling formula

Proof: Since $\Gamma(\bar{z}) = \overline{\Gamma(z)}$, we may suppose that $t \geq 1$. Further we may assume that t exceeds sufficiently large number depending only on a and b, otherwise Corollary follows. It suffices to prove (i), which implies immediately (ii), (iii) and (iv).

- By the previous theorem and the previous lemma, we have

$$\log \Gamma(z) = \frac{1}{2}\log(2\pi) + \left(\sigma + it - \frac{1}{2}\right)\log(\sigma + it) - (\sigma + it) + \frac{\theta}{8t}\arctan\frac{t}{\sigma}, \qquad \text{where} \quad |\theta| \leq 1.$$

- The second term on the right-hand side above is equal to

$$\left(\sigma - \frac{1}{2} + it\right)\left(\log\left(\sqrt{\sigma^2 + t^2}\right) + i\arctan\frac{t}{\sigma}\right).$$

- Then

$$\log\left(\sqrt{\sigma^2 + t^2}\right) = \log t + \frac{1}{2}\log\left(1 + \frac{\sigma^2}{t^2}\right) = \log t + O\left(\frac{1}{t^2}\right).$$

Stirling formula

- Also
$$\arctan\frac{t}{\sigma}=\frac{\pi}{2}-\arctan\frac{\sigma}{t}=\frac{\pi}{2}-\frac{\sigma}{t}+O\left(\frac{1}{t^3}\right).$$
- Further, we have
$$\frac{\theta}{8t}\arctan\frac{t}{\sigma}=\frac{\theta}{8t}\left(\frac{\pi}{2}-\arctan\frac{\sigma}{t}\right)=O\left(\frac{1}{t}\right)$$
- Therefore, we conclude
$$\begin{aligned}\log\Gamma(z)=&\frac{1}{2}\log(2\pi)+\frac{\pi i}{2}\left(\sigma-\frac{1}{2}\right)-\frac{\pi}{2}t+\left(\sigma-\frac{1}{2}\right)\log t\\&+it(\log t-1)+O\left(\frac{1}{t}\right).\end{aligned}$$
- Hence $\Gamma(z)=\sqrt{2\pi}e^{\frac{\pi i}{2}\left(\sigma-\frac{1}{2}\right)}e^{-\frac{\pi}{2}t}t^{\sigma-\frac{1}{2}}e^{it(\log t-1)}\left(1+O\left(\frac{1}{t}\right)\right).$ □

Stirling formula

Lemma

For $n \geq 0$, we have

(i) $(z+n)^{-2} = (z+n)^{-1} - (z+n+1)^{-1} + (z+n)^{-2}(z+n+1)^{-1}$.

(ii) *Further, we have*

$$(z+n)^{-2}(z+n+1)^{-1}$$
$$= \frac{1}{2}(z+n)^{-2} - \frac{1}{2}(z+n+1)^{-2} + \frac{1}{2}(z+n)^{-2}(z+n+1)^{-2}.$$

(iii) *We also have*

$$(z+n)^{-2}(z+n+1)^{-2}$$
$$= \frac{1}{3}(z+n)^{-3} - \frac{1}{3}(z+n+1)^{-3} - \frac{1}{3}(z+n)^{-3}(z+n+1)^{-3}.$$

Stirling formula

Proof of (i): We have

$$\begin{aligned}(z+n)^{-2} - (z+n)^{-2}(z+n+1)^{-1} &= (z+n)^{-2}\left(1-(z+n+1)^{-1}\right) \\ &= (z+n)^{-1}(z+n+1)^{-1} = (z+n)^{-1} - (z+n+1)^{-1}.\end{aligned}$$

□

Proof of (ii): We have

$$\begin{aligned}&(z+n)^{-2}(z+n+1)^{-1} - \frac{1}{2}(z+n)^{-2}(z+n+1)^{-2} \\ &\quad = \frac{1}{2}(z+n)^{-2}(z+n+1)^{-2}(2z+2n+1) \\ &\quad = \frac{1}{2}(z+n)^{-2}(z+n+1)^{-2}\left((z+n+1)^2-(z+n)^2\right) \\ &\quad = \frac{1}{2}(z+n)^{-2} - \frac{1}{2}(z+n+1)^{-2}.\end{aligned}$$

□

Stirling formula

Proof of (iii): By multiplying both sides of

$$\begin{aligned}&(z+n)^{-2}(z+n+1)^{-2}\\&=\frac{1}{3}(z+n)^{-3}-\frac{1}{3}(z+n+1)^{-3}-\frac{1}{3}(z+n)^{-3}(z+n+1)^{-3},\end{aligned}$$

by $3(z+n)^3(z+n+1)^3$, we prove that

$$3(z+n+1)(z+n)=(z+n+1)^3-(z+n)^3-1.$$

The right-hand side is equal to

$$\begin{aligned}&(z+n+1)^2+(z+n)^2+(z+n)(z+n+1)-1\\&=(z+n+2)(z+n)+(z+n)^2+(z+n+1)(z+n)\\&=3(z+n+1)(z+n). \qquad \square\end{aligned}$$

Stirling formula

Theorem

For $z > 0$, we have

$$\sqrt{\frac{2\pi}{z}}\left(\frac{z}{e}\right)^z \le \Gamma(z) \le \sqrt{\frac{2\pi}{z}}\left(\frac{z}{e}\right)^z e^{\frac{1}{12z}}.$$

Proof: For $z > 0$, we observe that the last inequality is equivalent to

$$0 \le \log\Gamma(z) - \left(z - \frac{1}{2}\right)\log z + z - \frac{1}{2}\log(2\pi) \le \frac{1}{12}z^{-1}.$$

by taking logarithms.

- Note that

$$\frac{d^2}{dz^2}\log\Gamma(z) = \frac{d}{dz}\left(\frac{\Gamma'(z)}{\Gamma(z)}\right) = \sum_{n=0}^{\infty}(z+n)^{-2} > 0.$$

Stirling formula

- By the previous Lemma (i), the last sum in is equal to

$$z^{-1} + \sum_{n=0}^{\infty} (z+n)^{-2}(z+n+1)^{-1}.$$

- Now we apply the last Lemma (ii) to each term of the above sum, and we obtain

$$\sum_{n=0}^{\infty} (z+n)^{-2}(z+n+1)^{-1} = \frac{1}{2}z^{-2} + \frac{1}{2}\sum_{n=0}^{\infty} (z+n)^{-2}(z+n+1)^{-1}.$$

- Thus

$$\frac{d^2}{dz^2}\log\Gamma(z) = z^{-1} + \frac{1}{2}z^{-2} + \frac{1}{2}\sum_{n=0}^{\infty} (z+n)^{-2}(z+n+1)^{-2},$$

and since $z > 0$, we conclude that

$$\frac{d^2}{dz^2}\log\Gamma(z) \geq z^{-1} + \frac{1}{2}z^{-2}.$$

Stirling formula

- Further, by the previous Lemma (iii), we have

$$\begin{aligned}\frac{d^2}{dz^2}\log\Gamma(z) &= z^{-1}+\frac{1}{2}z^{-2}+\frac{1}{6}z^{-3}-\frac{1}{6}\sum_{n=0}^{\infty}(z+n)^{-3}(z+n+1)^{-3}\\ &\le z^{-1}+\frac{1}{2}z^{-2}+\frac{1}{6}z^{-3}.\end{aligned}$$

- Combining the above two inequalities, we obtain

$$0\le \frac{d^2}{dz^2}\log\Gamma(z)-z^{-1}-\frac{1}{2}z^{-2}\le \frac{1}{6}z^{-3}.$$

Stirling formula

- Let
$$F(z) = \frac{d}{dz}\log\Gamma(z) - \log z + \frac{1}{2}z^{-1},$$
so that
$$0 \leq F'(z) \leq \frac{1}{6}z^{-3}.$$
- This implies that F is non-decreasing. Further, by integrating from z_0 to z with $z > z_0 > 1$, we have
$$0 \leq \int_{z_0}^{z} F'(\zeta)d\zeta \leq \frac{1}{6}\int_{z_0}^{z}\zeta^{-3}d\zeta.$$
- Thus
$$0 \leq F(z) - F(z_0) \leq \frac{1}{12}\left(z_0^{-2} - z^{-2}\right) \leq \frac{1}{12}z_0^{-2}.$$
- Therefore, $F(z)$ for $z > 1$ is bounded above and hence
$$\lim_{z\to\infty} F(z) = c$$
exists.

Stirling formula

- Further, by letting z tend to infinity and taking $z_0 = z$, we have

$$-\frac{1}{12}z^{-2} \leq F(z) - c \leq 0.$$

- Now we define

$$g(z) = \log\Gamma(z) - \left(z - \frac{1}{2}\right)\log z + z - cz \tag{6.13.15}$$

so that we see that

$$g'(z) = F(z) - c.$$

- Further we derive that

$$-\frac{1}{12}z^{-2} \leq g'(z) \leq 0,$$

which implies

$$\lim_{z\to\infty} g(z) = c_1$$

exists and $0 \leq g(z) - c_1 \leq \frac{1}{12}z^{-1}$.

Stirling formula

- Next we consider

$$g(z+1)-g(z)=-\left(z+\frac{1}{2}\right)\log\left(\frac{z+1}{z}\right)+1-c.$$

- By letting z tend to infinity on both sides, we derive that $c=0$.
- Now, we note that

$$\begin{aligned} g(2z)-g(z)-g\left(z+\frac{1}{2}\right) \\ &= \log\frac{\Gamma(2z)}{\Gamma(z)\Gamma\left(z+\frac{1}{2}\right)}-\left(2z-\frac{1}{2}\right)\log 2-\left(2z-\frac{1}{2}\right)\log z \\ &+\left(z-\frac{1}{2}\right)\log z+z\log\left(z+\frac{1}{2}\right)-\frac{1}{2} \\ &= \log\frac{2^{\frac{1}{2}-2z}\Gamma(2z)}{\Gamma(z)\Gamma\left(z+\frac{1}{2}\right)}+z\log\left(\frac{z+\frac{1}{2}}{z}\right)-\frac{1}{2}. \end{aligned}$$

Stirling formula

- By letting z tend to infinity on both sides, and using duplication formula
$$\Gamma(2z)\Gamma\left(1/2\right) = 2^{2z-1}\Gamma\left(z\right)\Gamma\left(z+1/2\right),$$
we conclude that
$$-c_1 = \lim_{z\to\infty}\left(g(2z) - g(z) - g\left(z+\frac{1}{2}\right)\right) = -\frac{1}{2}\log(2\pi),$$
and hence
$$c_1 = \frac{1}{2}\log(2\pi).$$
- This completes the proof of the theorem. □