Lecture 16

Harmonic functions

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Cauchy–Riemann equations

Theorem

Let $\Omega \subseteq \mathbb{C}$ be open and $f : \Omega \to \mathbb{C}$ be holomorphic. Then $\frac{\partial f}{\partial x} = \frac{1}{i}\frac{\partial f}{\partial y}$, where $\partial/\partial x$ and $\partial/\partial y$ denote the usual partial derivatives in the x and y variables respectively. If $f = u + iv$ for some real valued functions $u, v : \Omega \to \mathbb{C}$, then we have

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad \text{and} \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}. \tag{C-R}$$

These relations are called the **Cauchy–Riemann** *equations.*

Theorem

Suppose $f = u + iv$ is a complex-valued function defined on an open set Ω. If u and v are differentiable in the real sense and satisfy the Cauchy–Riemann equations (C-R) on Ω, then f is holomorphic on Ω.

Harmonic functions

Definition

Let $(x_0, y_0) \in \mathbb{R}^2$ and u be a real-valued function defined in a neighbourhood of (x_0, y_0). Then u is **harmonic** at (x_0, y_0) if

(i) u is continuous at (x_0, y_0).

(ii) u has continuous partial derivatives of the first and the second order at (x_0, y_0) satisfying

$$u_{xx}(x_0, y_0) + u_{yy}(x_0, y_0) = 0, \qquad (*)$$

where $u_{xy}(x_0, y_0) = \frac{\partial}{\partial y}\left(\frac{\partial u}{\partial x}(x_0, y_0)\right)$.

- The (*) is called **the Laplace equation**. Further u is called harmonic in Ω if it is harmonic at every point of Ω.

Harmonic functions

Remark

(i) We identify the elements (x, y) of $\mathbb{R}^2$ with $x + iy$ of $\mathbb{C}$ and it will be clear from the context whether we are taking (x, y) or $x + iy$.

(ii) For any $z = x + iy \in \mathbb{C}$ and any real-valued function $u = u(x, y)$ we write

$$u(z) = u(x, y).$$

(iii) If u is harmonic in Ω, then $u + c$ for any constant c is harmonic in Ω.

Theorem

Let $f \in H(\Omega)$ be given by

$$f(z) = u(x, y) + iv(x, y).$$

Then $\operatorname{Re}(f)$ *and* $\operatorname{Im}(f)$ *are harmonic in* Ω.

Harmonic functions

Proof: The proof depends on $f' \in H(\Omega)$ and $f'' \in H(\Omega)$.

- Let $f(z) = u(x, y) + iv(x, y)$ be given. First, we prove that u and v have continuous partial derivatives of orders $0, 1$ and 2 at every point of Ω. We prove the assertion for u and the proof for v is similar.
- Let $(x_0, y_0) \in \Omega$ and $z_0 = x_0 + iy_0$. Then we see that u is continuous at (x_0, y_0) since f is continuous at z_0.
- Further, by the Cauchy–Riemann equations, we have

$$f'(z_0) = u_x(x_0, y_0) + iv_x(x_0, y_0) = v_y(x_0, y_0) - iu_y(x_0, y_0).$$

- By differentiating this identity, we have u_x and u_y that are continuous at (x_0, y_0) since $f'(z)$ is continuous at z_0. Next, we have

$$\begin{aligned} f''(z_0) &= u_{xx}(x_0, y_0) + iv_{xx}(x_0, y_0) = v_{yx}(x_0, y_0) - iu_{yx}(x_0, y_0) \\ &= v_{xy}(x_0, y_0) - iu_{xy}(x_0, y_0) = -u_{yy}(x_0, y_0) - iv_{yy}(x_0, y_0). \end{aligned}$$

Harmonic functions

- This implies u has continuous partial derivative of order 2 at (x_0, y_0), since $f''(z)$ is continuous at z_0.
- Since (x_0, y_0) is an arbitrary point of Ω, we conclude that u has continuous partial derivatives of order $0, 1$ and 2 at every point of Ω.
- Differentiating the first (C-R) equation $u_x = v_y$ with respect to x and the second $v_x = -u_y$ with respect to y, we obtain

$$u_{xx}(x_0, y_0) = v_{yx}(x_0, y_0), \quad v_{xy}(x_0, y_0) = -u_{yy}(x_0, y_0),$$

which implies

$$u_{xx}(x_0, y_0) + v_{yy}(x_0, y_0) = 0,$$

since $v_{yx}(x_0, y_0) = v_{xy}(x_0, y_0)$. Hence u is harmonic in Ω. □

Identity theorem for harmonic functions

Theorem

Let u be harmonic in a region Ω and let V be a non-empty open subset of Ω such that $u = 0$ in V. Then $u = 0$ in Ω.

Proof: Let u be harmonic in Ω. For $z \in \Omega$ with $z = x + iy$, we consider

$$g(z) = u_x(x, y) - iu_y(x, y).$$

- We observe that u_x and $-u_y$ are defined in Ω and they satisfy Cauchy–Riemann equations in Ω since u is harmonic in Ω.
- Therefore, g is holomorphic in Ω.
- Further, $g = 0$ on V, since u_x and $-u_y$ vanish on V. Then $g = 0$ on Ω by identity theorem for holomorphic functions.
- Then $u_x = u_y = 0$ in Ω which implies that u is constant in Ω. □

Harmonic conjugate

Definition

Let u be harmonic in a region Ω. Then v is called a **harmonic conjugate** of u in Ω if

(i) v is harmonic in Ω.

(ii) There exists $f \in H(\Omega)$ such that

$$f = u + iv \quad \text{in} \quad \Omega.$$

Remark

- Let u be harmonic in a region Ω. Assume that v and v_1 are harmonic conjugates of u in Ω. Then there exist $f \in H(\Omega)$ and $f_1 \in H(\Omega)$ such that

$$f = u + iv, \quad f_1 = u + iv_1 \quad \text{in} \quad \Omega.$$

- Then $v - v_1 = -i\,(f - f_1) \in H(\Omega)$ is real valued. Therefore v and v_1 differ by a constant. Why?

Harmonic conjugate

Remark

- Let f be integrable on $[a, b]$ and

$$F(x) = \int_a^x f(t)dt \quad \text{for} \quad a \leq x \leq b.$$

  Then $F(x)$ is continuous in $[a, b]$. If f is continuous at $x_0 \in [a, b]$, then

$$F'(x_0) = f(x_0).$$

- Let f be integrable on $[a, b]$. If there exists a differentiable function F on $[a, b]$ such that $F' = f$. Then

$$\int_a^b f(t)dt = F(b) - F(a).$$

Harmonic conjugate

Theorem

Let $\Omega = D(0, R)$ where $0 < R \leq \infty$. Let u be harmonic in Ω. Then there exists a harmonic conjugate of u in Ω.

Proof: It suffices to find a real-valued function $v = v(x, y)$ satisfying:

(i) v has continuous partial derivatives at every point of Ω.

(ii) u and v satisfy the Cauchy–Riemann equations

$$u_x = v_y \quad \text{and} \quad u_y = -v_x$$

at every point of Ω.

- Then $f = u + iv \in H(\Omega)$. Now we see from the previous theorem that v will be a harmonic conjugate of u in Ω.

Harmonic conjugate

- For $(x, t) \in \Omega$, by the first equation in (ii), we have
$$u_x(x, t) = v_y(x, t).$$
- We integrate both sides with respect to t along a vertical line from 0 to y. We have
$$\int_0^y v_y(x, t)dt = \int_0^y u_x(x, t)dt.$$
- Thus
$$v(x, y) - v(x, 0) = \int_0^y u_x(x, t)dt.$$
- By putting $v(x, 0) = h(x)$, we have
$$v(x, y) = \int_0^y u_x(x, t)dt + h(x).$$
- We determine $h(x)$ such that the second equation in (ii) is satisfied.

Harmonic conjugate

- By substituting $v(x,y)$ in the second equation in (ii), we have

$$u_y(x,y) = -\frac{\partial}{\partial x}\int_0^y u_x(x,t)dt - h'(x) = -\int_0^y u_{xx}(x,t)dt - h'(x)$$
$$= \int_0^y u_{yy}(x,t)dt - h'(x) = u_y(x,y) - u_y(x,0) - h'(x).$$

- Therefore, $h'(x) = -u_y(x,0)$, which is satisfied if

$$h(x) = -\int_0^x u_y(s,0)ds + C,$$

where C is any constant. Then

$$v(x,y) = \int_0^y u_x(x,t)dt - \int_0^x u_y(s,0)ds + C.$$

- We check that v satisfies (i) and (ii) and hence v is a harmonic conjugate of u in Ω. □

Harmonic functions and simply connected regions

Theorem

A region Ω is simply connected if and only if every harmonic function in Ω has a harmonic conjugate in Ω.

Lemma

Let $u = u(x, y)$ and $v = v(x, y)$ be harmonic function in a region Ω. For $(x, y) \in \Omega$, let

$$R = R(x, y) = \frac{1}{2} \log \left((u(x, y))^2 + (v(x, y))^2 \right).$$

Then R is harmonic in Ω.

Proof: It is clear that R is continuous and it has continuous partial derivatives of orders 1 and 2 at every point of Ω.

- We show that R satisfies the Laplace equation at every point of Ω.

Harmonic functions and simply connected regions

- At $(x, y) \in \Omega$, we have

$$R_x = \frac{uu_x + vv_x}{u^2 + v^2}, \quad R_y = \frac{uu_y + vv_y}{u^2 + v^2},$$

and

$$\begin{aligned}\left(u^2 + v^2\right)^2 (R_{xx} + R_{yy}) = &\left(u^2 + v^2\right)\left(u_x^2 + v_x^2 + u_y^2 + v_y^2\right) \\ &- 2\left(uu_x + vv_x\right)^2 - 2\left(uu_y + vv_y\right)^2,\end{aligned}$$

by using $u_{xx} + u_{yy} = 0$ and $v_{xx} + v_{yy} = 0$.

- Simplifying, we obtain

$$\begin{aligned}\left(u^2 + v^2\right)^2 (R_{xx} + R_{yy}) = &u^2 v_x^2 + u^2 v_y^2 + v^2 u_x^2 + v^2 u_y^2 \\ &- \left(u^2 u_y^2 + u^2 u_x^2 + v^2 v_y^2 + v^2 u_x^2\right) \\ &- 2uvu_x u_y - 2uvv_x v_y = 0\end{aligned}$$

by using the Cauchy-Riemann equations. □

Harmonic functions and simply connected regions

Lemma

Let $\Omega = \mathbb{C}\backslash\{0\}$. For $z \in \Omega$ with $z = x + iy$, let

$$u(x, y) = \log|z| = \frac{1}{2}\log\left(x^2 + y^2\right).$$

Then u is harmonic in Ω.

Proof: We observe that u is continuous in Ω where it has continuous partial derivatives of orders 1 and 2, since

$$u_x = \frac{x}{x^2 + y^2}, \quad u_y = \frac{y}{x^2 + y^2}$$

and

$$u_{xx} = \frac{y^2 - x^2}{(x^2 + y^2)^2}, \quad u_{yy} = \frac{x^2 - y^2}{(x^2 + y^2)^2}.$$

The latter equation implies that u is harmonic in Ω. □

Harmonic functions and simply connected regions

Lemma

Let D_1 and Ω_1 be open discs. Let F be holomorphic function from D_1 into Ω_1 and u be harmonic in Ω_1. Then $u \circ F$ is harmonic in D_1.

Proof: Let $F(z) = A(x, y) + iB(x, y)$ for $z = x + iy \in D_1$.

- Since u is harmonic in Ω_1 and D_1 is a disc, then there exists $G \in H(\Omega_1)$ such that

$$G(z) = \phi(x, y) + i\psi(x, y) \quad \text{for} \quad z = x + iy \in \Omega_1,$$

where $\phi(x, y) = u(x, y)$. Then, for $z = x + iy \in D_1$, we have

$$\begin{aligned} G \circ F(z) &= G(A(x, y) + iB(x, y)) \\ &= \phi(A(x, y), B(x, y)) + i\psi(A(x, y), B(x, y)) \\ &= u(A(x, y), B(x, y)) + i\psi(A(x, y), B(x, y)), \end{aligned}$$

and $\mathrm{Re}(G \circ F(z)) = u(A(x, y), B(x, y)) = u \circ F(z)$. Now we conclude that $u \circ F$ is harmonic in D_1 and we are done. □

Harmonic functions and simply connected regions

Theorem

A region Ω is simply connected if and only if every harmonic function in Ω has a harmonic conjugate in Ω.

Proof: Assume that Ω is simply connected and let u be harmonic in Ω. We show that u has a harmonic conjugate in Ω.

- We may assume that $\Omega \neq \mathbb{C}$ otherwise the assertion follows from the previous theorem.
- Then, by the Riemann mapping theorem, there exists an analytic homeomorphism F from D onto Ω.
- In the previous lemma, we take $D_1 = D(z_0, s), \Omega_1 = D(F(z_0), r)$ and F is holomorphic function from D_1 into Ω_1. Since $\Omega_1 \subseteq \Omega$, we see that u is harmonic in Ω_1. Let $u \circ F = u_1$.

Harmonic functions and simply connected regions

- Let $z_0 \in D$. Then $F(z_0) \in \Omega$ and there exist $0 < s < r < 1$ such that

$$F(D(z_0, s)) \subseteq D(F(z_0), r) \subseteq \Omega.$$

- Then u_1 is harmonic in D_1 by the previous lemma. In particular, u_1 is harmonic at z_0.
- Since z_0 is an arbitrary point of D, we see that u_1 is harmonic in D. Hence, there exist v_1 harmonic in D and $f_1 \in H(D)$ such that

$$f_1 = u_1 + iv_1 \quad \text{in} \quad D.$$

- Then

$$f_1 \circ F^{-1} = u + iv_1 \circ F^{-1} \quad \text{in} \quad \Omega,$$

  and $f_1 \circ F^{-1} \in H(\Omega)$. Hence, we conclude that $v_1 \circ F^{-1}$ is harmonic conjugate of u in Ω.

Harmonic functions and simply connected regions

- Now, let Ω be a region and assume that every harmonic function in Ω has a harmonic conjugate in Ω. We show that Ω is simply connected.
- We may assume that $\Omega \neq \mathbb{C}$ otherwise the assertion follows since $\mathbb{C}$ is simply connected.
- It suffices to show that for every $f \in H(\Omega)$ with $\frac{1}{f} \in H(\Omega)$, there exists $g \in H(\Omega)$ such that $f(z) = g^2(z)$ for $z \in \Omega$. Then Ω is conformally equivalent to D. Hence Ω is simply connected as desired.
- Let $f \in H(\Omega)$ with $\frac{1}{f} \in H(\Omega)$. We set

$$\mathrm{Re}(f) = u, \quad \mathrm{Im}(f) = v.$$

- Then u and v are harmonic in Ω. For $x + iy \in \Omega$, we set

$$R(x,y) = \log|f(x+iy)| = \frac{1}{2}\log\left((u(x,y))^2 + (v(x,y))^2\right),$$

  which is defined since $f(z) \neq 0$ for $z \in \Omega$.
- $R(x,y)$ is harmonic in Ω as it was shown above.

Harmonic functions and simply connected regions

- Then, by our assumption, there exists a harmonic function S in Ω and $g_1 \in H(\Omega)$ such that
$$g_1 = R + iS \quad \text{in} \quad \Omega.$$
Let
$$h(z) = e^{g_1(z)} \quad \text{for} \quad z \in \Omega.$$
- Then $\frac{f(z)}{h(z)} \in H(\Omega)$ and
$$\left|\frac{f(z)}{h(z)}\right| = 1 \quad \text{for} \quad z \in \Omega.$$
- Therefore $\frac{f(z)}{h(z)}$ is constant in Ω by the open mapping theorem.
- Then $f(z) = ce^{g_1(z)} = e^{g_1(z)+c_1}$, where c and c_1 are constants.
- By putting
$$g(z) = e^{\frac{g_1(z)+c_1}{2}},$$
we see that $g(z) \in H(\Omega)$ and $f(z) = (g(z))^2$ for $z \in \Omega$. □

Mean Value Property (MVP) of harmonic functions

Definition

Let u be real-valued continuous function in a region Ω. Then u has **mean value property (MVP)** in Ω if for every $a \in \Omega$, we have

$$u(a) = \frac{1}{2\pi} \int_0^{2\pi} u\left(a + re^{i\theta}\right) d\theta,$$

whenever $\overline{D}(a, r) \subseteq \Omega$.

Theorem

Let u be harmonic in a region Ω. Then u satisfies MVP in Ω.

Proof Let $a \in \Omega$ with $\overline{D}(a, r) \subseteq \Omega$. There exists an open disc E such that

$$\overline{D}(a, r) \subseteq E \subseteq \Omega$$

and u has a harmonic conjugate in E.

Mean Value Property (MVP) of harmonic functions

- Therefore there exists $f \in H(E)$ such that $u = \mathrm{Re}(f)$. Now

$$f(a) = \frac{1}{2\pi i} \int_{|z-a|=r} \frac{f(z)}{z-a} dz$$

by the Cauchy integral formula.

- By putting $z - a = re^{i\theta}$ with $0 \leq \theta \leq 2\pi$, we have

$$f(a) = \frac{1}{2\pi i} \int_0^{2\pi} \frac{f\left(a + re^{i\theta}\right) ire^{i\theta}}{re^{i\theta}} d\theta = \frac{1}{2\pi} \int_0^{2\pi} f\left(a + re^{i\theta}\right) d\theta.$$

- By comparing the real parts on both the sides, we obtain

$$u(a) = \frac{1}{2\pi} \int_0^{2\pi} u\left(a + re^{i\theta}\right) d\theta.$$

- This holds for every $a \in \Omega$ whenever $\overline{D}(a, r) \subseteq \Omega$. □

Maximum principle for the continuous functions with MVP

Theorem

Let u be real-valued continuous function in a region Ω and assume that u has MVP in Ω. Suppose that there exists $a \in \Omega$ such that

$$u(z) \leq u(a) \quad \text{for all} \quad z \in \Omega.$$

Then u is constant in Ω.

Proof: We assume that u is not constant in Ω. Let u be continuous in a region satisfying MVP in Ω and there exists $a \in \Omega$ such that $u(z) \leq u(a)$ for $z \in \Omega$. We consider

$$A = \{z \in \Omega : u(z) = u(a)\}.$$

- We may assume that $A \neq \emptyset$, since $a \in A$. It suffices to show that A is both open and closed.
- Then $A = \Omega$, since Ω is connected and hence u is constant in Ω.

Maximum principle for the continuous functions with MVP

- Let $z \in \overline{A}$. Then there exists a sequence $(z_n)_{n\in\mathbb{N}} \subseteq A$ such that $\lim_{n\to\infty} z_n = z$. Since u is continuous, we have $\lim_{n\to\infty} u(z_n) = u(z)$. But $u(z_n) = a$ for $n \geq 1$ since $z_n \in A$. Therefore $u(z) = u(a)$ which implies that $z \in A$. Thus $\overline{A} \subseteq A$ and hence A is closed.
- Now we show that A is open. Let $z_0 \in A$ and there exists $r > 0$ with $D(z_0, r) \subseteq \Omega$ such that $D(z_0, r)$ is not contained in A.
- Then there exists $b \in D(z_0, r)$ and $b \notin A$. Thus

$$u(b) < u(a) = u(z_0)$$

- Since u is continuous, there exists $s > 0$ such that

$$u(z) < u(a) \quad \text{for} \quad z \in D(b, s).$$

- Let $|b - z_0| = \rho < r$. Then there exists an arc on the circle $|z - z_0| = \rho$ containing b of positive length where $u(z) < u(z_0)$ and $u(z) \leq u(a) = u(z_0)$ elsewhere on the circle.

Maximum principle for the continuous functions with MVP

- Therefore
$$\frac{1}{2\pi}\int_0^{2\pi} u\left(z_0+\rho e^{i\theta}\right)d\theta < u\left(z_0\right).$$
- On the other hand, we have
$$\frac{1}{2\pi}\int_0^{2\pi} u\left(z_0+\rho e^{i\theta}\right)d\theta = u\left(z_0\right),$$

since u satisfies *MVP* by the assumption. This is a contradiction. □

Corollary

Let Ω be a bounded region. Assume that u is a non-constant real-valued continuous function defined on $\overline{\Omega}$ and u has MVP in Ω. Then there exists $a\in\partial\Omega$ such that
$$u(z) < u(a) \quad \text{for} \quad z\in\Omega.$$

Proof: Prove it!

Poisson kernel

Definition

For $0 \leq r < 1$ and $0 \leq \theta \leq 2\pi$, the function

$$P_r(\theta) = \sum_{n=-\infty}^{\infty} r^{|n|} e^{in\theta} \qquad (*)$$

is called the **Poisson kernel**.

- We understand that $0^0 = 1$ in the sum on the right-hand side of (*) so that $P_r(\theta) = 1$ if $r = 0$.

Lemma

For $0 \leq r < 1$ *and* $0 \leq \theta \leq 2\pi$, *we have*

$$P_r(\theta) = \mathrm{Re}\left(\frac{1 + re^{i\theta}}{1 - re^{i\theta}}\right) = \frac{1 - r^2}{1 - 2r\cos\theta + r^2}. \qquad (**)$$

Poisson kernel

Proof: For $0 \leq |z| < 1$, we have

$$\begin{aligned}\frac{1+z}{1-z} &= (1+z)(1-z)^{-1} \\ &= (1+z)\left(1+z+z^2+\cdots\right) \\ &= 1+2\sum_{n=1}^{\infty} z^n.\end{aligned}$$

Here the rearrangement of terms of the series is permissible since the series is absolutely convergent.

- By putting $z = re^{i\theta}$ with $0 \leq r < 1$ above, we have

$$\frac{1+re^{i\theta}}{1-re^{i\theta}} = 1+2\sum_{n=1}^{\infty} r^n e^{in\theta}.$$

Poisson kernel

- Now

$$\begin{aligned}
\operatorname{Re}\left(\frac{1+re^{i\theta}}{1-re^{i\theta}}\right) &= 1+2\sum_{n=1}^{\infty} r^n \cos n\theta \\
&= 1+\sum_{n=1}^{\infty} r^n \left(e^{in\theta}+e^{-in\theta}\right) \\
&= 1+\sum_{n=1}^{\infty} r^n e^{in\theta} + \sum_{n=-\infty}^{-1} r^{|n|} e^{in\theta} \\
&= 1+\sum_{\substack{n=-\infty \\ n\neq 0}}^{\infty} r^{|n|} e^{in\theta} = P_r(\theta).
\end{aligned}$$

Poisson kernel

- Further

$$\frac{1+re^{i\theta}}{1-re^{i\theta}} = \frac{\left(1+re^{i\theta}\right)\left(1-re^{-i\theta}\right)}{\left|1-re^{i\theta}\right|^2} = \frac{1-r^2+2ir\sin\theta}{\left|1-re^{i\theta}\right|^2}$$

and

$$\left|1-re^{i\theta}\right|^2 = 1-2r\cos\theta+r^2.$$

Therefore

$$P_r(\theta) = \operatorname{Re}\left(\frac{1+re^{i\theta}}{1-re^{i\theta}}\right) = \frac{1-r^2}{1-2r\cos\theta+r^2}.$$

Poisson kernel

Lemma

(a) *For* $0 \leq r < 1$, *we have* $P_r(\theta) > 0$ *for* $0 \leq \theta \leq 2\pi$ *and* $P_r(\theta)$ *is periodic with period* 2π. *Further*

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} P_r(\theta)d\theta = 1.$$

(b) *Let* $\delta > 0$. *Then*

$$\lim_{r\to 1^-} P_r(\theta) = 0$$

uniformly in θ *with* $\delta \leq |\theta| \leq \pi$.

Proof of (a): It is clear that $P_r(\theta) > 0$ for $0 \leq \theta \leq 2\pi$ and periodic with period 2π, since

$$P_r(\theta) = \operatorname{Re}\left(\frac{1+re^{i\theta}}{1-re^{i\theta}}\right) = \frac{1-r^2}{1-2r\cos\theta + r^2}.$$

Poisson kernel

- By integrating both sides of the previous identity, we obtain

$$\int_{-\pi}^{\pi} P_r(\theta) = \int_{-\pi}^{\pi} \sum_{n=-\infty}^{\infty} r^{|n|} e^{in\theta} d\theta = \sum_{n=-\infty}^{\infty} r^{|n|} \int_{-\pi}^{\pi} e^{in\theta} d\theta = 2\pi,$$

since the series converges uniformly in θ and

$$\int_{-\pi}^{\pi} e^{in\theta} = \begin{cases} 2\pi & \text{if } n = 0, \\ 0 & \text{if } n \neq 0. \end{cases} \qquad \square$$

Proof of (b): Let $\delta > 0$ and $0 < r < 1$.

- We may assume that $|\theta| \leq \frac{\pi}{2}$ otherwise the assertion follows immediately from the formula

$$P_r(\theta) = \operatorname{Re}\left(\frac{1 + re^{i\theta}}{1 - re^{i\theta}}\right) = \frac{1 - r^2}{1 - 2r\cos\theta + r^2}.$$

Poisson kernel

- By differentiating both sides with respect to θ in the previous formula and setting $\theta = t$, we have

$$P_r'(t) = \frac{-\left(1-r^2\right)2r\sin t}{\left(1-2r\cos t + r^2\right)^2}.$$

- Then

$$P_r'(t) < 0 \quad \text{for} \quad \delta \le t \le \frac{\pi}{2}.$$

- Thus

$$P_r(\theta) \le P_r(\delta) \quad \text{for} \quad \delta \le \theta \le \frac{\pi}{2}.$$

- Since $P_r(\theta) = P_r(-\theta)$, we obtain

$$P_r(\theta) \le P_r(\delta) \quad \text{for} \quad \delta \le |\theta| \le \frac{\pi}{2}.$$

- Since $\lim_{r\to 1^-} P_r(\delta) = 0$, we derive that $\lim_{r\to 1^-} P_r(\theta) = 0$ uniformly in $\delta \le |\theta| \le \frac{\pi}{2}$. This completes the proof. □

Dirichlet problem for open discs

Theorem

Let $a \in \mathbb{C}$, $\rho > 0$ and f be real-valued continuous function defined on the circle $C(a, \rho)$. Then there exists unique real-valued continuous function u in $\overline{D}(a, \rho)$ such that u is harmonic in $D(a, \rho)$ and

$$u(z) = f(z) \quad \text{for} \quad z \in C(a, \rho).$$

Proof: We claim that there is no loss of generality in assuming that $a = 0$ and $\rho = 1$. Indeed, suppose that the assertion of the theorem is valid with $a = 0$ and $\rho = 1$. Let f be real-valued continuous function on $C(a, \rho)$.

- Then we consider

$$g(z) = f(a + \rho z) \quad \text{for} \quad |z| = 1.$$

- We note that g is continuous on $|z| = 1$. Then there is a real-valued continuous function $v(z)$ in $\overline{D}$ and harmonic in D such that

$$v(z) = g(z) \quad \text{for} \quad |z| = 1.$$

Dirichlet problem for open discs

- Let
$$u(z) = v\left(\frac{z-a}{\rho}\right) \quad \text{for} \quad z \in \overline{D}(a,\rho).$$
- Then the conclusion follows, since u is a real-valued continuous function in $\overline{D}(a,\rho)$ and harmonic in $D(a,\rho)$ and such that $u(z) = f(z)$ for $|z-a| = \rho$.
- Let $M = \max\left\{\left|f\left(e^{i\phi}\right)\right| : |\phi| \le 2\pi\right\}$. We prove the theorem with
$$u\big(re^{i\theta}\big) = \begin{cases} \frac{1}{2\pi}\int_{-\pi}^{\pi} P_r(\theta-\phi) f\left(e^{i\phi}\right) d\phi & \text{if } 0 \le r < 1, 0 \le \theta \le 2\pi, \\ f\left(e^{i\theta}\right) & \text{if } r = 1, 0 \le \theta \le 2\pi. \end{cases}$$
- Let $0 \le r < 1$. We show that u is real part of an analytic function and then it is harmonic in D.

Dirichlet problem for open discs

- By the formula $P_r(\theta) = \operatorname{Re}\left(\frac{1+re^{i\theta}}{1-re^{i\theta}}\right) = \frac{1-r^2}{1-2r\cos\theta+r^2}$, we have

$$u\left(re^{i\theta}\right) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f\left(e^{i\phi}\right) \operatorname{Re}\left(\frac{1+re^{i(\theta-\phi)}}{1-re^{i(\theta-\phi)}}\right) d\phi.$$

- We observe that

$$u\left(re^{i\theta}\right) = \operatorname{Re}(g(z)) \quad \text{with} \quad z = re^{i\theta},$$

  where

$$g(z) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f\left(e^{i\phi}\right)\left(\frac{e^{i\phi}+z}{e^{i\phi}-z}\right) dz,$$

  which is analytic in D.
- Therefore u is harmonic in D, hence it is continuous in D. Further $u\left(e^{i\theta}\right) = f\left(e^{i\theta}\right)$ for $0 \leq \theta \leq 2\pi$.
- Now we show that u is continuous on $|z| = 1$.

Dirichlet problem for open discs

- We have $\left|u\left(e^{i\theta}\right)\right| = \left|f\left(e^{i\theta}\right)\right| \leq M$ for $0 \leq \theta \leq 2\pi$.
- Further $f\left(e^{i\theta}\right)$ with $0 \leq \theta \leq 2\pi$ is uniformly continuous. Therefore for $\varepsilon > 0$, there exists $\delta > 0$ such that

$$\left|u\left(e^{i\theta}\right) - u\left(e^{i\phi}\right)\right| = \left|f\left(e^{i\theta}\right) - f\left(e^{i\phi}\right)\right| < \varepsilon,$$

  whenever $|\theta - \phi| \leq \delta$. Let A be an arc of the circle $|z| = 1$ with $e^{i\theta}$ as the centre of the arc and subtending an angle δ at the origin. Then $|\theta - \phi| \leq \delta$ whenever $e^{i\phi} \in A$.
- Thus it suffices to show that for any $e^{i\theta}$ with $0 \leq \theta \leq 2\pi$, we have

$$\left|u\left(re^{i\theta}\right) - u\left(e^{i\theta}\right)\right| < 2\varepsilon \quad \text{whenever} \quad r \to 1^-.$$

- We also have

$$u\left(re^{i\theta}\right) = \frac{1}{2\pi}\int_{-\pi}^{\pi} P_r\left(\theta - \gamma\right) u\left(e^{i\gamma}\right) d\gamma \quad \text{for} \quad 0 \leq r < 1.$$

Dirichlet problem for open discs

- By setting $\theta - \gamma = \phi$, we obtain for $0 \le r < 1$ that
$$u\big(re^{i\theta}\big) = \frac{1}{2\pi}\int_{-\pi+\theta}^{\pi+\theta} P_r(\phi)u\big(e^{i(\theta-\phi)}\big)\,d\phi = \frac{1}{2\pi}\int_{-\pi}^{\pi} P_r(\phi)u\big(e^{i(\theta-\phi)}\big)\,d\phi,$$
since the integrand is periodic with period 2π.
- We further observe that
$$\begin{aligned} u\big(re^{i\theta}\big) - u\big(e^{i\theta}\big) =& \frac{1}{2\pi}\int_{-\pi}^{\pi} P_r(\phi)\Big(u\big(e^{i(\theta-\phi)}\big) - u\big(e^{i\theta}\big)\Big)\,d\phi \\ =& \frac{1}{2\pi}\int_{|\phi|<\delta} P_r(\phi)\Big(u\big(e^{i(\theta-\phi)}\big) - u\big(e^{i\theta}\big)\Big)\,d\phi \\ &+ \frac{1}{2\pi}\int_{\pi\ge|\phi|\ge\delta} P_r(\phi)\Big(u\big(e^{i(\theta-\phi)}\big) - u\big(e^{i\theta}\big)\Big)\,d\phi. \end{aligned}$$
- Since $P_r(\phi) > 0$, the absolute value of the first integral is at most
$$\frac{\varepsilon}{2\pi}\int_{-\pi}^{\pi} P_r(\phi)\,d\phi = \varepsilon.$$

Dirichlet problem for open discs

- The absolute value of the second integral is at most

$$2M \max_{\delta \leq |\phi| \leq \pi} P_r(\phi) < 2M \frac{\varepsilon}{2M}$$

when $r \to 1^-$, hence

$$\left| u\left(re^{i\theta}\right) - u\left(e^{i\theta}\right) \right| < 2\varepsilon \quad \text{whenever} \quad r \to 1^-.$$

- It remains to show that u is unique satisfying the assertion of the theorem. Let v be a continuous function in $\overline{D}$ such that v is harmonic in D and $v(z) = f(z)$ for $|z| = 1$.
- Now we consider the function $w = u - v$. Then w is harmonic in D, and therefore it has (MVP) in D. Since $w = 0$ on $|z| = 1$, we conclude by the maximum principle, that $w = 0$ in $\overline{D}$. Why?
- Hence $v = u$. The proof is completed. □

Dirichlet problem for open discs

Theorem

Let u be a real-valued continuous function with (MVP) in a region. Then u is harmonic in Ω.

Proof: Let u be a real-valued continuous function with (MVP) in Ω.

- Let $a \in \Omega$. Since Ω is open, there exists $\rho > 0$ such that $\overline{D}(a, \rho) \subseteq \Omega$. It suffices to show that u is harmonic in $D(a, \rho)$. Then u is harmonic at a and the assertion follows since a is an arbitrary point in Ω.
- Since $\overline{D}(a, \rho) \subseteq \Omega$, we see that u is continuous in $\overline{D}(a, \rho)$ and it has (MVP) in $D(a, \rho)$. By the Dirichlet problem, there exists a real-valued continuous function v in $\overline{D}(a, \rho)$ such that v is harmonic in $D(a, \rho)$ and such that

$$u(z) = v(z) \quad \text{if} \quad |z - a| = \rho.$$

- Further v has (MVP) in Ω, since v is harmonic. Next we consider

$$g = u - v \quad \text{in} \quad \overline{D}(a, \rho).$$

Dirichlet problem for open discs

- We observe that g is real-valued continuous function in $\overline{D}(a, \rho)$ and it has MVP in $D(a, \rho)$.
- Further
$$g(z) = 0 \quad \text{if} \quad |z - a| = \rho.$$
- Assume that g is not a constant function. Then $g(z) < 0$ in $D(a, \rho)$ by the maximum principle and $g(z) > 0$ in $D(a, \rho)$ by the minimum principle. This is a contradiction.
- Therefore g is a constant function c in $D(a, \rho)$. In fact $c = 0$ since g is continuous in $\overline{D}(a, \rho)$ and zero on $|z - a| = \rho$.
- Hence $u = v$ is harmonic in $D(a, \rho)$ as desired. □