Lecture 13

Conformal mappings

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Conformal mappings

Definition

Let Ω_1, Ω_2 and Ω be open subsets of $\mathbb{C}$.

- A one-to-one holomorphic mapping f from Ω_1 onto Ω_2 such that $f^{-1} \in H(\Omega_2)$ is called an **analytic homeomorphism** of Ω_1 onto Ω_2. If $\Omega = \Omega_1 = \Omega_2$, then f is called an **analytic automorphism** of Ω.
- A one-to-one holomorphic mapping from Ω_1 onto Ω_2 is called a **conformal mapping** of Ω_1 onto Ω_2. If $\Omega = \Omega_1 = \Omega_2$, then f is called a **conformal mapping** of Ω.

Recall the inverse mapping theorem:

Theorem

Suppose that $\Omega \subseteq \mathbb{C}$ is open, $f \in H(\Omega)$, and f is one-to-one in Ω. Then $f'(z) \neq 0$ for every $z \in \Omega$, and the inverse of f is holomorphic.

Conformal mappings

Remarks

- By the inverse mapping theorem, we see that a conformal mapping of Ω_1 onto Ω_2 is an analytic homeomorphism of Ω_1 onto Ω_2, and a conformal mapping of Ω is an automorphism of Ω.
- We say that Ω_1 and Ω_2 are **conformally equivalent** whenever there is a conformal mapping f from Ω_1 onto Ω_2 (or between Ω_1 and Ω_2) and we write $\Omega_1 \sim \Omega_2$, where $\sim$ is an equivalence relation.
- If $\Omega_1 \sim \Omega_2$, then the corresponding conformal mapping $f : \Omega_1 \to \Omega_2$ satisfies $f'(z) \neq 0$ for $z \in \Omega_1$ by the inverse mapping theorem.
- Some authors take the condition $f'(z) \neq 0$ for $z \in \Omega_1$ as the definition of a conformal mapping.
- The latter condition is less restrictive. For example, $f(z) = z^2$ is not one-to-one but f' vanishes nowhere in $\mathbb{C}\backslash\{0\}$.

Conformal mappings

Remarks

- There is a geometric consequence of the condition $f'(z) \neq 0$ and it is at the root of this discrepency of terminology in the definitions. A holomorphic map that satisfies this condition preserves angles.
- Loosely speaking, if two curves γ and η intersect at z_0, and α is the oriented angle between the tangent vectors to these curves, then the image curves $f \circ \gamma$ and $f \circ \eta$ intersect at $f(z_0)$, and their tangent vectors form the same angle α.
- We will always mean, as is the general practice is complex analysis that conformal mapping is one-to-one onto holomorphic function.
- The set of all automorphisms of Ω is a group under composition of mappings and we denote it by $\mathrm{Aut}(\Omega)$.
- For $\lambda \in \mathbb{C}$ with $|\lambda| = 1$, we observe that $z \mapsto \lambda z$ is an automorphism of $D = D(0,1)$ and this automorphism is called a **rotation**.

Upper half-plane and the unit disc

Theorem

Let $\Omega_1 = \mathbb{H} = \{z \in \mathbb{C} : \operatorname{Im} z > 0\}$ *and* $\Omega_2 = D = \{z \in \mathbb{C} : |z| < 1\}$. *Then the map* $F : \Omega_1 \to \Omega_2$, *given by*

$$F(z) = \frac{i - z}{i + z} \quad \text{for} \quad z \in \Omega_1,$$

is a conformal map with the inverse $G : \Omega_2 \to \Omega_1$ *given by*

$$G(w) = i\frac{1 - w}{1 + w} \quad \text{for} \quad w \in \Omega_2.$$

In other words, Ω_1 *and* Ω_2 *are conformally equivalent.*

Upper half-plane and the unit disc

Proof: Recall that $F(z) = \frac{i-z}{i+z}$ for $z \in \Omega_1$.

- F is one-to-one. Indeed, if $F(u) = F(v)$, then

$$\frac{i-u}{i+u} = \frac{i-w}{i+w} \quad \Longleftrightarrow \quad 2iu = 2iw.$$

- Further we observe that F is analytic in Ω_1 since $-i \notin \Omega_1$.
- Let $z \in \Omega_1$ and write $z = x + iy$ with $y > 0$. Then

$$F(z) = \frac{-x - i(y-1)}{x + i(y+1)}.$$

- Thus

$$|F(z)|^2 = F(z)\overline{F(z)} = \frac{x^2 + (y-1)^2}{x^2 + (y+1)^2} < 1,$$

  and hence $F(z) \in \Omega_2$ since $y > 0$.

Upper half-plane and the unit disc

- It remains to show that F is onto. Indeed, let $w \in \Omega_2$ and we write $w = u + iv$ with $u^2 + v^2 < 1$.
- We find $z \in \Omega_1$ such that $F(z) = w$. Thus

$$\frac{i - z}{i + z} = w,$$

and then

$$z = i\left(\frac{1 - w}{1 + w}\right) = i\left(\frac{1 - u - iv}{1 + u + iv}\right) = i\frac{(1 - u - iv)(1 + u - iv)}{(1 + u + iv)(1 + u - iv)}.$$

- Therefore

$$\text{Im}(z) = \frac{1 - u^2 - v^2}{(1 + u)^2 + v^2} > 0,$$

and hence $z \in \Omega_1$. □

Upper half-plane and a sector

Theorem

For $n \in \mathbb{N}$, let

$$\Omega_1 = \left\{ z \in \mathbb{C} : 0 < \arg z < \frac{\pi}{n} \right\}$$

and $\Omega_2 = \mathbb{H}$. We show that Ω_1 and Ω_2 are conformally equivalent.

Proof: For this, consider holomorphic function

$$f(z) = z^n \quad \text{for} \quad z \in \Omega_1.$$

- We write $z = re^{i\theta}$ with $0 < \theta < \frac{\pi}{n}$, then $f(z) = r^n e^{in\theta}$.
- Thus f is a function from Ω_1 into Ω_2, since

$$\text{Im}(f(z)) = r^n \sin n\theta > 0 \quad \text{for} \quad 0 < \theta < \frac{\pi}{n}.$$

- f is one-to-one. Indeed, let $z_1, z_2 \in \Omega_1$ with $f(z_1) = f(z_2)$ and

$$z_1 = r_1 e^{i\theta_1}, z_2 = r_2 e^{i\theta_2} \quad \text{with} \quad 0 < \theta_1 \leq \theta_2 < \frac{\pi}{n}.$$

Upper half-plane and a sector

- Then
$$r_1^n e^{in\theta_1} = r_2^n e^{in\theta_2},$$
implying
$$\left(\frac{r_1}{r_2}\right)^n = e^{in(\theta_2-\theta_1)} = \cos\left(n\left(\theta_2-\theta_1\right)\right) + i\sin\left(n\left(\theta_2-\theta_1\right)\right).$$
- By comparing the imaginary parts on both sides, we obtain that $\sin\left(n\left(\theta_2-\theta_1\right)\right) = 0$ implying $\theta_1 = \theta_2$ since $0 \le n\left(\theta_2-\theta_1\right) < \pi$.
- Hence $r_1 = r_2$ and then $z_1 = z_2$, proving that f is one-to-one.
- Now, we show that f is onto. Let $w = Re^{i\phi} \in \Omega_2$. Then $0 < \phi < \pi$.
- Setting $w^{1/n}$ by taking the principal value of logarithm, we see that
$$w^{1/n} = e^{\frac{1}{n}\log w} = e^{\frac{1}{n}\log R + i\frac{\phi}{n}} \in \Omega_1,$$
and $f\left(w^{1/n}\right) = w$.
- Hence Ω_1 and Ω_2 are conformally equivalent. □

Upper half-plane and the positive quadrant of $\mathbb{C}$

Theorem

Let $\Omega_1 = \left\{x + iy : y > 0, x^2 + y^2 < 1\right\}$, *be the upper half disc and* $\Omega_2 = \{u + iv : u > 0, v > 0\}$ *be the positive quadrant of* $\mathbb{C}$. *Then the mapping*

$$f(z) = \frac{1+z}{1-z}$$

is a conformal mapping of Ω_1 *onto* Ω_2.

Proof: Let $z = x + iy \in \Omega_1$. Then $|x| < 1, y > 0, x^2 + y^2 < 1$ and

$$\begin{aligned} f(z) = \frac{1+x+iy}{1-x-iy} &= \frac{(1+x+iy)(1-x+iy)}{(1-x-iy)(1-x+iy)} \\ &= \frac{1-\left(x^2+y^2\right)}{(1-x)^2+y^2} + i\frac{2y}{(1-x)^2+y^2}. \end{aligned}$$

Since $\mathrm{Re}(f(z)) > 0$ and $\mathrm{Im}(f(z)) > 0$, we see that $f(z) \in \Omega_2$.

Upper half-plane and the positive quadrant of $\mathbb{C}$

- It is clear that f is analytic in Ω_1 since $1 \notin \Omega_1$.
- f is one-to-one, since $f(z) = \frac{1+z}{1-z} = \frac{1+w}{1-w} = f(w)$ implies $2z = 2w$.
- It remains to show that f is onto. Let $w = u + iv \in \Omega_2$. Then $u > 0, v > 0$ and we shall find $z \in \Omega_1$ such that $f(z) = w$.
- Thus $\frac{1+z}{1-z} = w$ and then

$$
\begin{aligned}
z = \frac{w-1}{w+1} = \frac{(u-1)+iv}{(u+1)+iv} &= \frac{((u-1)+iv)((u+1)-iv)}{((u+1)+iv)((u+1)-iv)} \\
&= \frac{u^2+v^2-1}{(u+1)^2+v^2} + i\frac{2v}{(u+1)^2+v^2}.
\end{aligned}
$$

- Thus $\text{Im}(z) > 0$ since $v > 0$. Further $|z| < 1$ since $|w-1| < |w+1|$ whenever w lies in the first quadrant and hence $z \in \Omega_1$. □

Upper half-plane and a strip

Theorem

Let $\Omega_1 = \mathbb{H}$ be the upper half-plane and $\Omega_2 = \{u + iv : 0 < v < \pi\}$ be an open strip. Then Ω_1 and Ω_2 are conformally equivalent.

Proof: Let us consider

$$f(z) = \log z, \quad \text{for} \quad z \in \Omega_1,$$

where the branch of logarithm is principal.

- Then $f \in H(\Omega_1)$ and one-to-one. We show that f is onto.
- Let $w = u + iv \in \Omega_2$. Then $0 < v < \pi$ and we take

$$z = e^w = e^u e^{iv} = e^u(\cos v + i \sin v).$$

- Thus

$$\operatorname{Im}(z) = e^u \sin v > 0$$

  since $0 < v < \pi$. Therefore $z \in \Omega_1$ and $f(z) = w$.
- Hence f is a conformal mapping between Ω_1 and Ω_2. □

Schwarz lemma

Theorem

Let f be holomorphic in the open unit disc D.

- *If f satisfies $|f(z)| \leq 1$, and $f(0) = 0$, then $|f(z)| \leq |z|$ for $|z| < 1$ and $|f'(0)| \leq 1$.*
- *If $|f(z)| = |z|$ for some $z \in D$ or $|f'(0)| = 1$, then $f(z) = cz$ in D for some constant c whose absolute value is 1.*

Proof: Let

$$g(z) = \begin{cases} \frac{f(z)}{z}, & \text{if } z \neq 0, \\ f'(0), & \text{if } z = 0. \end{cases}$$

- Then g is analytic in $|z| < 1$ since $f(0) = 0$.
- For $0 \leq r < 1$, we derive from the maximum modulus theorem that

$$\max_{|z| \leq r} |g(z)| = \max_{|z|=r} |g(z)| = \frac{1}{r} \left(\max_{|z|=r} |f(z)| \right) \leq \frac{1}{r}.$$

Schwarz lemma

- Letting r tend to 1, we get

$$|g(z)| \leq 1 \quad \text{in} \quad |z| < 1.$$

- Hence,

$$|f(z)| \leq |z| \quad \text{in} \quad |z| < 1.$$

- Assume that either $|f(z_0)| = |z_0|$ for some $|z_0| < 1$ or $|f'(0)| = 1$.
- Then $g(z) = 1$ for some $|z| < 1$. Therefore g is constant in D of absolute value 1 by the maximum principle.
- Then $f(z) = cz$ in $|z| < 1$, where c is a constant with $|c| = 1$. □

Definition

For $a \in D$, we consider a function $\phi_a : \mathbb{C}_\infty \to \mathbb{C}_\infty$ given by

$$\phi_a(z) = \frac{z - a}{1 - \bar{a}z}. \qquad (*)$$

Schwarz lemma and its consequences

- We observe that

$$\phi_a\left(\frac{1}{\overline{a}}\right) = \infty, \quad \phi_{-a}\left(-\frac{1}{\overline{a}}\right) = \infty,$$
$$\phi_a(\infty) = -\frac{1}{\overline{a}}, \qquad \phi_{-a}(\infty) = \frac{1}{\overline{a}}.$$

Lemma

Let $a \in D$ and $\mathbb{T} = \{z \in \mathbb{C} : |z| = 1\}$ be the torus.

(i) ϕ_a *is one-to-one, onto and the inverse of* ϕ_a *is* ϕ_{-a} *and* $\phi_a(a) = 0$.

(ii) ϕ_a *is analytic in* $D\left(0, \frac{1}{\overline{a}}\right)$ *containing* $\overline{D}(0,1)$.

(iii) $\phi_a(\mathbb{T}) = \mathbb{T}$.

(iv) $\phi_a(D) = D$.

(v) $\phi_a'(0) = 1 - |a|^2$, *and* $\phi_a'(a) = \frac{1}{1-|a|^2}$.

Schwarz lemma and its consequences

Proof (i): We show

$$\phi_{-a} \circ \phi_a(z) = z, \quad \text{and} \quad \phi_a \circ \phi_{-a}(z) = z \quad \text{for} \quad z \in \mathbb{C}.$$

We prove the first and the proof for the latter is similar.

- If $z = \infty$, then

$$\phi_{-a} \circ \phi_a(\infty) = \phi_{-a}\left(-\frac{1}{\overline{a}}\right) = \infty.$$

- If $z = \frac{1}{\overline{a}}$, then

$$\phi_{-a} \circ \phi_a\left(\frac{1}{\overline{a}}\right) = \phi_{-a}(\infty) = \frac{1}{\overline{a}}.$$

- Therefore we may assume that $z \neq \frac{1}{\overline{a}}$ and $z \neq \infty$. Now

$$\phi_{-a} \circ \phi_a(z) = \phi_{-a}\left(\frac{z-a}{1-\overline{a}z}\right) = \frac{\frac{z-a}{1-\overline{a}z} + a}{1 + \overline{a}\frac{z-a}{1-\overline{a}z}} = \frac{z(1-a\overline{a})}{1-a\overline{a}} = z,$$

since $a\overline{a} = |a|^2 < 1$ for $a \in D$. □

Schwarz lemma and its consequences

Proof (ii): Note that $\phi_a(z)$ is analytic in $\mathbb{C}$ except at $z = \frac{1}{\bar{a}}$ with $\frac{1}{|\bar{a}|} > 1$.

- Let $1 < r < \frac{1}{|\bar{a}|}$, then $\phi_a(z)$ is analytic in $D(0, r)$.
- Since $\overline{D} \subseteq D(0, r)$, the assertion follows. □

Proof (iii): For $t \in \mathbb{R}$, we have

$$\phi_a\left(e^{it}\right) = \frac{e^{it} - a}{1 - \bar{a}e^{it}} = \frac{e^{it} - a}{e^{it}\left(e^{-it} - \bar{a}\right)}.$$

- Therefore
$$\left|\phi_a\left(e^{it}\right)\right| = 1.$$
- Thus $\phi_a(\mathbb{T}) \subseteq \mathbb{T}$. Similarly $\phi_{-a}(\mathbb{T}) \subseteq \mathbb{T}$, implying
$$\mathbb{T} \subseteq \phi_a(\mathbb{T}).$$

  Hence $\phi_a(\mathbb{T}) = \mathbb{T}$. □

Proof (iv): It follows from (iii) by the maximum modulus principle. □

Schwarz lemma and its consequences

Proof (v): We have

$$\phi_a'(z) = \frac{(1-\bar{a}z) - (z-a)(-\bar{a})}{(1-\bar{a}z)^2}$$
$$= \frac{1-a\bar{a}}{(1-\bar{a}z)^2} = \frac{1-|a|^2}{(1-\bar{a}z)^2}.$$

- Thus
$$\phi_a'(0) = 1-|a|^2,$$
and
$$\phi_a'(a) = \frac{1-|a|^2}{(1-|a|^2)^2} = \frac{1}{1-|a|^2}.$$
- This completes the proof the lemma. □

Schwarz lemma and its consequences

Lemma

Let f be non-constant and analytic in D, and satisfy $|f(z)| < 1$ for $z \in D$. Let $w \in D$ with $f(w) = a$. Then

$$\left|f'(w)\right| \leq \frac{1 - |a|^2}{1 - |w|^2}.$$

Moreover equality occurs only when

$$f = \phi_{-a} \circ (c\phi_w) \text{ in } D,$$

for some constant c whose absolute value is 1.

Proof: We consider

$$g(z) = \phi_a \circ f \circ \phi_{-w}(z) \quad \text{in} \quad |z| < 1.$$

Schwarz lemma and its consequences

- Then by the previous lemma (i), (ii) and (iv), we see that $g(z)$ is analytic in $|z| < 1$ satisfying $|g(z)| < 1$ for $|z| < 1$ and

$$g(0) = \phi_a \circ f \circ \phi_{-w}(0) = \phi_a \circ f(w) = \phi_a(a) = 0.$$

- Thus the assumptions of Schwarz lemma are satisfied and hence we conclude that

$$\left|g'(0)\right| \leq 1.$$

- Now we compute $g'(0)$ by using the previous lemma (v):

$$\begin{aligned} g'(0) &= (\phi_a \circ f)' \left(\phi_{-w}(0)\right) \phi'_{-w}(0) \\ &= \left(1 - |w|^2\right) (\phi_a \circ f)' (w) \\ &= \left(1 - |w|^2\right) \phi'_a(f(w)) f'(w) \\ &= \left(1 - |w|^2\right) \phi'_a(a) f'(w) \\ &= \frac{1 - |w|^2}{1 - |a|^2} f'(w). \end{aligned}$$

Schwarz lemma and its consequences

- By $|g'(0)| \leq 1$, we obtain

$$\left|f'(w)\right| \leq \frac{1-|a|^2}{1-|w|^2}.$$

- Suppose that we have equality above. Then $|g'(0)| = 1$.
- Now we derive from Schwarz lemma that there exists a constant c with $|c| = 1$ such that

$$g(z) = cz \quad \text{for} \quad |z| < 1.$$

- Therefore

$$\phi_a \circ f \circ \phi_{-w}(z) = \chi_c(z) \quad \text{for} \quad |z| < 1,$$

where $\chi_c(z) = cz$. Thus

$$f = \phi_{-a} \circ \chi_c \circ \phi_w = \phi_{-a} \circ (c\phi_w) \text{ in } D. \qquad \square$$

Disc automorphisms

Theorem

Let f be an automorphism of D and $w \in D$ such that $f(w) = 0$. Then

$$f = c\phi_w \text{ in } D$$

where c is a constant of absolute value 1.

Remark

We observe that rotation is an automorphism of D. On the other hand, the theorem with $w = 0$ yields that all the automorphisms of D carrying the centre to centre are given by rotations.

Proof of the theorem: Let h be the inverse of f. Then

$$h(f(z)) = z \quad \text{for} \quad z \in D, \quad \text{and} \quad h(0) = w.$$

Disc automorphisms

- By the inverse mapping theorem, $h \in H(D)$ and $h'(z) \neq 0$ for $z \in D$.
- By differentiating both sides of $h(f(z)) = z$, we obtain

$$h'(f(z))f'(z) = 1 \quad \text{for} \quad z \in D.$$

- Setting $z = w$, we have $h'(f(w))f'(w) = h'(0)f'(w) = 1$.
- Now we derive from the previous lemma with $a = 0$ that

$$\left|f'(w)\right| \leq \frac{1}{1 - |w|^2}, \qquad \left|h'(0)\right| \leq 1 - |w|^2,$$

and hence

$$\left|f'(w)\right| = \frac{1}{1 - |w|^2}$$

- By the previous lemma applied again, we conclude that

$$f = \phi_0 \circ c\phi_w = c\phi_w \text{ in } D$$

where c is a constant of absolute value 1. □

Automorphisms of the upper half-plane

Definition

Denote by $G = \mathrm{SL}_2(\mathbb{R})$ the set of all 2×2 matrices

$$M = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

with $a, b, c, d \in \mathbb{R}$ such that its determinant $ad - bc$ is equal to 1.

- It is clear that G is a group under matrix multiplication.
- For $M \in G$, we denote

$$f_M(z) = \frac{az + b}{cz + d}.$$

- For $M \in G$ and $N \in G$, we check by direct computation that

$$f_{MN} = f_M \circ f_N.$$

Automorphisms of the upper half-plane

Lemma (A)

Let $M \in G$. Then $f_M \in \mathrm{Aut}(\mathbb{H})$.

Proof: Fix $M \in G$, then it is clear that f_M is analytic in $\mathbb{H}$ since $-\frac{d}{c} \notin \mathbb{H}$.

- Let $z \in \mathbb{H}$ with $z = x + iy$. Then $y > 0$ and

$$\begin{aligned}
\mathrm{Im}\left(f_M(z)\right) &= \mathrm{Im}\left(\frac{az+b}{cz+d}\right) = \mathrm{Im}\left(\frac{ax+b+iay}{cx+d+icy}\right) \\
&= \mathrm{Im}\left(\frac{(ax+b+iya)(cx+d-iyc)}{(cx+d)^2+c^2y^2}\right) \\
&= \frac{(cx+d)ya-(ax+b)yc}{(cx+d)^2+c^2y^2} \\
&= \frac{y}{(cx+d)^2+c^2y^2} > 0,
\end{aligned}$$

since $ad - bc = 1$. Thus f_M is analytic function of $\mathbb{H}$ into $\mathbb{H}$.

Automorphisms of the upper half-plane

- Let $f_M(z_1) = f_M(z_2)$. Then $(ad - bc)z_1 = (ad - bc)z_2$ implying $z_1 = z_2$, since $ad - bc = 1$. Therefore f_M is one-to-one as desired.
- We show that f_M is onto. Let $N \in G$ be the inverse of M. Then

$$N = \begin{pmatrix} d & -b \\ -c & a \end{pmatrix},$$

and we check that for $z \in \mathbb{H}$ we have

$$f_M\left(\frac{dz - b}{-cz + a}\right) = f_M(f_N(z)) = \frac{a\frac{dz-b}{-cz+a} + b}{c\frac{dz-b}{-cz+a} + d} = z$$

since $ad - bc = 1$. □

Automorphisms of the upper half-plane

Lemma (B)

Let $z, w \in \mathbb{H}$. Then there exists $M \in G$ such that $f_M(z) = w$.

Proof: Let $z = x + iy \in \mathbb{H}$, then $y > 0$. Let b and $c \neq 0$ be real numbers to be specified later.

- Set $M = M_2 M_1$, where

$$M_1 = \begin{pmatrix} 0 & -c^{-1} \\ c & 0 \end{pmatrix}, \qquad M_2 = \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix}.$$

- It is clear that $M_1, M_2 \in G$ and thus $M \in G$.
- Then, we have

$$f_M(z) = f_{M_2} \circ f_{M_1}(z) = f_{M_2}\left(-\frac{1}{c^2 z}\right).$$

Automorphisms of the upper half-plane

- We observe that
$$\operatorname{Im}\left(-\frac{1}{c^2 z}\right) = \frac{y}{c^2|z|^2} = 1,$$
by choosing c suitably. Further we write
$$-\frac{1}{c^2 z} = \chi + i, \quad \text{where} \quad \chi \in \mathbb{R}.$$
- Then, we also see that
$$f_M(z) = f_{M_2}(\chi + i) = \chi + b + i$$
- Choosing $b = -\chi$ we obtain
$$f_M(z) = i.$$
- Let $w \in \mathbb{H}$. Similarly, as above, there exists $M_3 \in G$ such that $f_{M_3}(w) = i$. Let $M_4 = M_3^{-1}M$. Then $M_4 \in G$ and
$$f_{M_4}(z) = f_{M_3^{-1}}(f_M(z)) = f_{M_3^{-1}}(i) = w. \qquad \square$$

Automorphisms of the upper half-plane

Lemma (C)

For $\theta \in \mathbb{R}$, *let*

$$M_\theta = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix},$$

and

$$F(z) = \frac{i - z}{i + z} \quad \textit{for} \quad z \in \mathbb{H}.$$

Then

$$F \circ f_{M_\theta} \circ F^{-1}(z) = e^{-2i\theta} z \quad \textit{for} \quad z \in D.$$

Proof: We know that F is a conformal mapping of $\mathbb{H}$ onto D.

- For $\theta \in \mathbb{R}$, let
$$\lambda_\theta(z) = e^{-2i\theta} z \quad \text{for} \quad z \in D.$$
- It suffices to show that
$$F \circ f_{M_\theta} = \lambda_\theta \circ F \text{ in } \mathbb{H}.$$

Automorphisms of the upper half-plane

- Indeed, if $F \circ f_{M_\theta} = \lambda_\theta \circ F$, then for $z \in D$, we have

$$\begin{aligned} F \circ f_{M_\theta} \circ F^{-1}(z) &= F \circ f_{M_\theta}\left(F^{-1}(z)\right) \\ &= \lambda_\theta \circ F\left(F^{-1}(z)\right) = \lambda_\theta(z) = e^{-2i\theta} z. \end{aligned}$$

- Now for $z \in \mathbb{H}$ we have

$$\begin{aligned} F \circ f_{M_\theta}(z) = F\left(f_{M_\theta}(z)\right) &= F\left(\frac{z\cos\theta - \sin\theta}{z\sin\theta + \cos\theta}\right) \\ &= \frac{i - \frac{z\cos\theta - \sin\theta}{z\sin\theta + \cos\theta}}{i + \frac{z\cos\theta - \sin\theta}{z\sin\theta + \cos\theta}} \\ &= \frac{z(i\sin\theta - \cos\theta) + (i\cos\theta + \sin\theta)}{z(i\sin\theta + \cos\theta) + (i\cos\theta - \sin\theta)} \\ &= \frac{-ze^{-i\theta} + ie^{-i\theta}}{ze^{i\theta} + ie^{i\theta}} = e^{-2i\theta}\frac{i - z}{i + z} = e^{-2i\theta}F(z). \end{aligned}$$

□

Automorphisms of the upper half-plane

Theorem

We have

$$\mathrm{Aut}(\mathbb{H}) = \{f_M : M \in G\}.$$

Proof: By Lemma (A) we have $\{f_M : M \in G\} \subseteq \mathrm{Aut}(\mathbb{H})$ it suffices to show that $\mathrm{Aut}(\mathbb{H}) \subseteq \{f_M : M \in G\}$.

- Let $f \in \mathrm{Aut}(\mathbb{H})$. Then there exists $\beta \in \mathbb{H}$ such that $f(\beta) = i$.
- By Lemma (B) there exists $N \in G$ such that $f_N(i) = \beta$. We set

$$g = f \circ f_N.$$

- Then $g(i) = f\left(f_N(i)\right) = f(\beta) = i$. Now we consider $F \circ g \circ F^{-1}$, where $F(z) = \frac{i-z}{i+z}$. We observe that $F \circ g \circ F^{-1} \in \mathrm{Aut}(D)$ and

$$F \circ g \circ F^{-1}(0) = F(g(i)) = F(i) = 0.$$

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- Hence we conclude that there exists $\theta \in \mathbb{R}$ so that

$$F \circ g \circ F^{-1} = t_\theta \text{ in } D,$$

where

$$t_\theta(z) = e^{-2i\theta} z \quad \text{for} \quad z \in D.$$

- Further, we see from Lemma (C) that

$$F \circ f_{M_\theta} \circ F^{-1} = t_\theta \text{ in } D.$$

- Thus we conclude that

$$f \circ f_N = g = f_{M_\theta}.$$

- Hence

$$f = f_{M_\theta N^{-1}}$$

and $M_\theta N^{-1} \in G$. □