Lecture 11

Maximum principle in unbounded regions and the Phragmen–Lindelöf method

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Maximum modulus principle

Definition

Let f be defined on Ω and $a \in \Omega$. Then $|f|$ has a **local maximum** at a if there exists $\delta > 0$ such that $D(a, \delta) \subseteq \Omega$ and $|f(a)| \geq |f(z)|$ for every $z \in D(a, \delta)$. Further, we say that $|f|$ has no local maximum in Ω if $|f|$ does not have local maximum at every point of Ω. Similarly, we define a **local minimum**.

Theorem

Suppose that Ω is a region and $f \in H(\Omega)$.

(a) *Then $|f|$ has no local maximum at any point of Ω, unless f is constant.*

(b) *Moreover, if the closure of Ω is compact and f is continuous on $\overline{\Omega}$, then*

$$\sup_{z \in \Omega} |f(z)| \leq \sup_{z \in \partial\Omega} |f(z)|.$$

The boundedness of Ω in the previous theorem is necessary

Remark

- Thus the maximum modulus principle is **not valid in unbounded** open sets. We elaborate it by the following example. Let

$$\Omega = \left\{ z = x + iy : -\frac{\pi}{2} < y < \frac{\pi}{2} \right\}$$

and

$$f(z) = \exp(\exp(z)).$$

- Then Ω is unbounded and $\max_{z \in \partial\Omega} |f(z)| = 1$, since

$$\left| f\left(x \pm \frac{\pi}{2} i \right) \right| = \left| \exp\left(\exp\left(x \pm \frac{\pi}{2} i \right) \right) \right| = |\exp(\pm i \exp(x))| = 1.$$

- On the other hand, $f(x) = e^{e^x} \to \infty$ as x tends to infinity through positive reals.

Phragmen–Lindelöf method

- It has already been pointed out that the maximum modulus principle need not be valid in unbounded regions.
- We will illustrate the Phragmen–Lindelöf method, which will enable to us to prove that a function is constant in a region whenever we control its growth in the region.
- We need not necessarily assume that a function is bounded in the region for concluding that it is constant as is the case of the Liouville theorem.
- For example, an entire function $f(z)$ is constant if

$$|f(z)| \leq 1 + |z|^{\frac{1}{2}}$$

for $z \in \mathbb{C}$.

Phragmen–Lindelöf method

Theorem

For given $a, b \in \mathbb{R}$, *let*

$$\Omega = \{(x + iy) : a < x < b\}.$$

Let f *be continuous on* $\overline{\Omega}$ *and* $f \in H(\Omega)$ *and assume that* $|f(z)| < B$ *for* $z \in \Omega$ *and fixed* $B > 0$. *For* $a \leq x \leq b$, *let*

$$M(x) = \sup\{|f(x + iy)| : -\infty < y < \infty\}.$$

Then we have

$$(M(x))^{b-a} \leq (M(a))^{b-x}(M(b))^{x-a} \quad \text{for} \quad a \leq x \leq b. \tag{*}$$

Phragmen–Lindelöf method

Proof: For $\varepsilon > 0$, we consider $f + \varepsilon$ in place of f if $M(a) = 0$ and $f - \varepsilon$ in place of f if $M(b) = 0$ and let ε tend to zero to observe that there is no loss of generality in assuming that $M(a) > 0$ and $M(b) > 0$.

- For $z \in \overline{\Omega}$, we write $z = x + iy$ with $a \leq x \leq b$.

Claim

Suppose that the theorem is valid for all functions f satisfying the assumptions of the theorem together with $M(a) = M(b) = 1$.

- We now prove the general case. Let

$$g(z) = (M(a))^{\frac{b-z}{b-a}} (M(b))^{\frac{z-a}{b-a}}.$$

- We observe that $g(z)$ is analytic in $\mathbb{C}$ and it has no zero in $\mathbb{C}$. Further

$$|g(z)| = (M(a))^{\frac{b-x}{b-a}} (M(b))^{\frac{x-a}{b-a}},$$

  and the exponents on the right-hand side above lie in $[0, 1]$.

Phragmen–Lindelöf method

- Therefore
$$|g(z)| \geq \tau \quad \text{for} \quad z \in \overline{\Omega}$$
where
$$\tau = \min(1, M(a)) \min(1, M(b)) > 0,$$
and
$$|g(a+iy)| = M(a), \quad \text{and} \quad |g(b+iy)| = M(b).$$
- Now we consider $h(z) = \frac{f(z)}{g(z)}$. By the definition of h and our assumption $M(a) = M(b) = 1$, we see that
$$\sup_{-\infty<y<\infty} |h(a+iy)| = \sup_{-\infty<y<\infty} |h(b+iy)| = 1,$$
and
$$|h(z)| \leq B\tau^{-1} \quad \text{for} \quad z \in \Omega.$$
- Therefore h satisfies the assumptions of the theorem. Hence
$$|h(x+iy)| \leq 1 \quad \text{for} \quad a \leq x \leq b.$$

Phragmen–Lindelöf method

- Using $|g(z)| = (M(a))^{\frac{b-x}{b-a}}(M(b))^{\frac{x-a}{b-a}}$ and $|h(x+iy)| \le 1$, we have
$$|f(x+iy)| \le |g(x+iy)| = (M(a))^{\frac{b-x}{b-a}}(M(b))^{\frac{x-a}{b-a}} \quad \text{for} \quad a \le x \le b,$$
which implies (*).
- It remains to prove the theorem when $M(a) = M(b) = 1$, which we assume from now on and we complete the proof by showing that $M(x) \le 1$ for $a \le x \le b$.
- Since f is continuous on $\overline{\Omega}$ and $|f(z)| < B$ for $z \in \Omega$, we see that
$$|f(z)| \le B \quad \text{for} \quad z \in \overline{\Omega}.$$
- For $\varepsilon > 0$, we define
$$h_\varepsilon(z) = \frac{1}{1 + \varepsilon(z-a)}.$$
- For a fixed $z \in \mathbb{C}$, we see that $\lim_{\varepsilon \to 0} h_\varepsilon(z) = 1$.

Phragmen–Lindelöf method

- Therefore it suffices to prove for every $\varepsilon > 0$ we have

$$|f(z)h_\varepsilon(z)| \le 1 \quad \text{for} \quad z \in \overline{\Omega}.$$

- First, we estimate $h_\varepsilon(z)$. We observe that

$$\mathrm{Re}(1 + \varepsilon(z - a)) \ge 1 \quad \text{for} \quad z \in \overline{\Omega}.$$

- Therefore

$$|h_\varepsilon(z)| \le \frac{1}{\mathrm{Re}(1 + \varepsilon(z - a))} \le 1 \quad \text{for} \quad z \in \overline{\Omega}. \tag{A}$$

- Next

$$|\,\mathrm{Im}(1 + \varepsilon(z - a))| \ge \varepsilon|y| \quad \text{for} \quad z \in \overline{\Omega},$$

and therefore

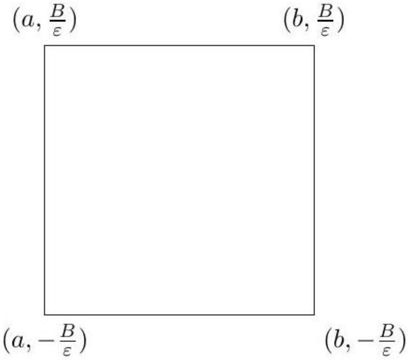
$$|f(z)h_\varepsilon(z)| \le \frac{B}{\varepsilon|y|} \quad \text{for} \quad z \in \overline{\Omega}. \tag{B}$$

Phragmen–Lindelöf method

- In particular

$$|f(z)h_\varepsilon(z)| \leq 1 \quad \text{if} \quad z \in \overline{\Omega} \quad \text{and} \quad |\operatorname{Im}(z)| \geq \frac{B}{\varepsilon}.$$

- We consider the closed set S consisting of rectangle as here



Phragmen–Lindelöf method

- Since $M(a) = M(b) = 1$, we see from (A) and (B) that

$$|f(z)h_\varepsilon(z)| \leq 1 \text{ for } z \in \partial S$$

- Therefore by the maximum principle, we have

$$|f(z)h_\varepsilon(z)| \leq 1 \text{ for } z \in S.$$

- Also by (B) we conclude that

$$|f(z)h_\varepsilon(z)| \leq 1 \text{ for } z \in \overline{\Omega} \backslash S.$$

- Hence

$$|f(z)h_\varepsilon(z)| \leq 1 \text{ for } z \in \overline{\Omega}.$$

This proves the theorem. □

Phragmen–Lindelöf method

Corollary

Suppose that the assumptions of the previous theorem are satisfied. Further suppose that f is not constant. Then

$$|f(z)| < \max(M(a), M(b)) \quad \textit{for} \quad z \in \Omega.$$

Proof: Assume that $M(a) \neq M(b)$. By the previous theorem, we have

$$M(x)^{b-a} < (\max(M(a), M(b)))^{b-a} \quad \text{for} \quad a \leq x \leq b.$$

Therefore, the assertion follows, since

$$M(x) < \max(M(a), M(b)) \quad \text{for} \quad a \leq x \leq b.$$

Let $M(a) = M(b)$. Then, by the previous theorem as above, we have $M(x) \leq M(a)$ for $a \leq x \leq b$. We fix arbitrary $z_0 \in \Omega$. Then there is $r > 0$ such that $\overline{D}(z_0, r) \subset \Omega$. By the maximum principle, we have

$$|f(z_0)| < \max_{|z-z_0|=r} |f(z)| \leq M(x) \leq M(a) \quad \text{for some} \quad a < x < b. \qquad \square$$

Hadamard three-circle theorem

Theorem

Let $0 < R_1 < R_2 < \infty$ *and*

$$A(0; R_1, R_2) = \{z \in \mathbb{C} : R_1 < |z| < R_2\}.$$

Let $g(z)$ *be continuous in* $\overline{A}(0; R_1, R_2)$ *and holomorphic in* $A(0; R_1, R_2)$*. Let*

$$B(R) = \max_{|z|=R} |g(z)| \quad \text{for} \quad R_1 \le R \le R_2.$$

Then

$$(B(R))^{\log R_2 - \log R_1} \le (B(R_1))^{\log R_2 - \log R} (B(R_2))^{\log R - \log R_1}$$

for $R_1 \le R \le R_2$*.*

Hadamard three-circle theorem

Proof: Let $\Omega = \{x + iy \in \mathbb{C} : \log R_1 < x < \log R_2\}$ and let $e(z) = e^z$.

- We observe that $e(z) \in \overline{A}(0; R_1, R_2)$ for $z \in \overline{\Omega}$. Thus e is a function from $\overline{\Omega}$ into $\overline{A}(0; R_1, R_2)$. Further it is onto.
- Moreover it maps a vertical line in $\overline{\Omega}$ passing through $(x, 0)$ onto a circle $\{z \in \mathbb{C} : |z| = e^x\}$ in $\overline{A}(0; R_1, R_2)$.
- Next we write $f(z) = g(e(z))$ for $z \in \overline{\Omega}$.
- We observe that f is continuous in $\overline{\Omega}$ and analytic in Ω. Since $\overline{A}(0; R_1, R_2)$ is compact, we see that f is bounded on $\overline{\Omega}$.
- Thus f satisfies all the assumptions of the previous theorem. Further for $R_1 \leq R \leq R_2$, we have

$$\begin{aligned} M(\log R) &= \sup\{|f(\log R + iy)| : -\infty < y < \infty\} \\ &= \sup\left\{g\left(Re^{i\theta}\right) : 0 \leq \theta \leq 2\pi\right\} = B(R). \end{aligned}$$

Hadamard three-circle theorem

- Hence we conclude from the previous theorem with $a = \log R_1$ and $b = \log R_2$ that

$$(M(\log R))^{\log R_2 - \log R_1} \le (M(\log R_1))^{\log R_2 - \log R} (M(\log R_2))^{\log R - \log R_1}.$$

- This together with

$$\begin{aligned} M(\log R) &= \sup\{|f(\log R + iy)| : -\infty < y < \infty\} \\ &= \sup\left\{g\left(Re^{i\theta}\right) : 0 \le \theta \le 2\pi\right\} = B(R), \end{aligned}$$

  implies that

$$(B(R))^{\log R_2 - \log R_1} \le (B(R_1))^{\log R_2 - \log R} (B(R_2))^{\log R - \log R_1}$$

  for $R_1 \le R \le R_2$.

- This completes the proof of Hadamard's theorem. □

Phragmen–Lindelöf method: applications

Theorem

Suppose

$$\Omega = \left\{ x + iy \in \mathbb{C} : |y| < \frac{\pi}{2} \right\}.$$

Suppose f is continuous on $\overline{\Omega}$, and $f \in H(\Omega)$, and there are constants $\alpha < 1$, $A < \infty$, such that

$$|f(z)| < \exp\{A \exp(\alpha |x|)\}, \quad \textit{whenever} \quad z = x + iy \in \Omega,$$

and

$$\left| f\left(x \pm \frac{\pi i}{2} \right) \right| \leq 1 \quad \textit{for} \quad -\infty < x < \infty.$$

Then $|f(z)| \leq 1$ for all $z \in \Omega$.

- Note that the conclusion does not follow if $\alpha = 1$, as is shown by the function $\exp(\exp z)$.

Phragmen–Lindelöf method: applications

Proof: Choose $\beta > 0$ so that $\alpha < \beta < 1$. For $\varepsilon > 0$, define

$$h_\varepsilon(z) = \exp\left\{-\varepsilon\left(e^{\beta z} + e^{-\beta z}\right)\right\}.$$

- For $z \in \overline{\Omega}$, we have

$$\mathrm{Re}\left[e^{\beta z} + e^{-\beta z}\right] = \left(e^{\beta x} + e^{-\beta x}\right)\cos\beta y \geq \delta\left(e^{\beta x} + e^{-\beta x}\right),$$

  where $\delta = \cos(\beta\pi/2) > 0$, since $|\beta| < 1$. Hence

$$|h_\varepsilon(z)| \leq \exp\left\{-\varepsilon\delta\left(e^{\beta x} + e^{-\beta x}\right)\right\} < 1 \quad \text{for} \quad z \in \overline{\Omega}.$$

- It follows that $|fh_\varepsilon| \leq 1$ on $\partial\Omega$ and that

$$|f(z)h_\varepsilon(z)| \leq \exp\left\{Ae^{\alpha|x|} - \varepsilon\delta\left(e^{\beta x} + e^{-\beta x}\right)\right\} \quad \text{for} \quad z \in \overline{\Omega}.$$

Phragmen–Lindelöf method: applications

- Fix $\varepsilon > 0$. Since $\varepsilon\delta > 0$ and $\beta > \alpha$, we have

$$\lim_{x \to \pm\infty} \left(Ae^{\alpha|x|} - \varepsilon\delta \left(e^{\beta x} + e^{-\beta x} \right) \right) = -\infty$$

- Hence there exists an x_0 so that

$$|f(z)h_\varepsilon(z)| \leq 1 \quad \text{for all} \quad x > x_0.$$

- Since $|fh_\varepsilon| \leq 1$ on the boundary of the rectangle whose vertices are $\pm x_0 \pm (\pi i/2)$, the maximum modulus theorem shows that actually $|fh_\varepsilon| \leq 1$ on this rectangle.
- Thus $|fh_\varepsilon| \leq 1$ at every point of Ω, for every $\varepsilon > 0$.

$$\lim_{\varepsilon \to 0} h_\varepsilon(z) = 1 \quad \text{for every} \quad z \in \mathbb{C},$$

so we conclude that $|f(z)| \leq 1$ for all $z \in \Omega$ as desired. □

Riesz interpolation theorem

Theorem

Let (X, μ) and (Y, v) be two σ-finite measure spaces. Let T be a linear operator defined on the set of all finitely simple functions on X and taking values in the set of measurable functions on Y. Let $1 \leq p_0, p_1, q_0, q_1 \leq \infty$ and assume that

$$\|T(f)\|_{L^{q_0}} \leq M_0 \|f\|_{L^{p_0}}, \quad \text{and} \quad \|T(f)\|_{L^{q_1}} \leq M_1 \|f\|_{L^{p_1}},$$

for all finitely simple functions f on X. Then for all $0 < \theta < 1$ we have

$$\|T(f)\|_{L^q} \leq M_0^{1-\theta} M_1^{\theta} \|f\|_{L^p} \tag{*}$$

for all finitely simple functions f on X, where

$$\frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1} \quad \text{and} \quad \frac{1}{q} = \frac{1-\theta}{q_0} + \frac{\theta}{q_1}. \tag{**}$$

Riesz interpolation theorem

Remarks

- Recall that a simple function is called finitely simple if it is supported in a set of finite measure. Finitely simple functions are dense in $L^p(X,\mu)$ for $0 < p < \infty$, whenever (X,μ) is a σ-finite measure space.
- Consequently, when $p < \infty$, by density, T has a unique bounded extension from $L^p(X,\mu)$ to $L^q(Y,v)$ when p and q are as in (**).
- The proof will be based on the following lemma:

Lemma

Let F be analytic in the open strip $S = \{z \in \mathbb{C} : 0 < \operatorname{Re} z < 1\}$, continuous and bounded on its closure, such that $|F(z)| \leq B_0$ when $\operatorname{Re} z = 0$ and $|F(z)| \leq B_1$ when $\operatorname{Re} z = 1$, for some $0 < B_0, B_1 < \infty$. Then $|F(z)| \leq B_0^{1-\theta} B_1^{\theta}$ when $\operatorname{Re} z = \theta$, for any $0 \leq \theta \leq 1$.

Proof: Exercise! □

Riesz interpolation theorem

Proof: Let

$$f = \sum_{k=1}^{m} a_k e^{i\alpha_k} \chi_{A_k}$$

be a finitely simple function on X, where $a_k > 0, \alpha_k$ are real, and A_k are pairwise disjoint subsets of X with finite measure.

- We need to control

$$\|T(f)\|_{L^q(Y,v)} = \sup_g \left| \int_Y T(f)(y) g(y) dv(y) \right|,$$

  where the supremum is taken over all finitely simple functions g on Y with $L^{q'}$ norm less than or equal to 1.

- Write

$$g = \sum_{j=1}^{n} b_j e^{i\beta_j} \chi_{B_j},$$

  where $b_j > 0, \beta_j$ are real, and B_j are pairwise disjoint subsets of Y with finite v measure.

Riesz interpolation theorem

- Let

$$P(z) = \frac{p}{p_0}(1-z) + \frac{p}{p_1}z \quad \text{and} \quad Q(z) = \frac{q'}{q'_0}(1-z) + \frac{q'}{q'_1}z.$$

- For z in the closed strip $\overline{S} = \{z \in \mathbb{C} : 0 \leq \operatorname{Re} z \leq 1\}$, define

$$f_z = \sum_{k=1}^{m} a_k^{P(z)} e^{i\alpha_k} \chi_{A_k}, \quad \text{and} \quad g_z = \sum_{j=1}^{n} b_j^{Q(z)} e^{i\beta_j} \chi_{B_j},$$

  and

$$F(z) = \int_Y T(f_z)(y) g_z(y) dv(y)$$

- Notice that $f_\theta = f$ and $g_\theta = f$. By linearity we have

$$F(z) = \sum_{k=1}^{m} \sum_{j=1}^{n} a_k^{P(z)} b_j^{Q(z)} e^{i\alpha_k} e^{i\beta_j} \int_Y T(\chi_{A_k})(y) \chi_{B_j}(y) dv(y)$$

Riesz interpolation theorem

- Since $a_k, b_j > 0, F$ is analytic in z, and the expression

$$\int_Y T\left(\chi_{A_k}\right)(y)\chi_{B_j}(y)dv(y)$$

is a finite constant, being an absolutely convergent integral; this is seen by Hölder's inequality with exponents q_0 and q_0' (or q_1 and q_1') and our assumption

$$\|T(f)\|_{L^{q_0}} \leq M_0\|f\|_{L^{p_0}}, \quad \text{and} \quad \|T(f)\|_{L^{q_1}} \leq M_1\|f\|_{L^{p_1}}.$$

- By the disjointness of the sets A_k we have (even when $p_0 = \infty$)

$$\|f_{it}\|_{L^{p_0}} = \|f\|_{L^p}^{\frac{p}{p_0}}$$

since $\left|a_k^{P(it)}\right| = a_k^{\frac{p}{p_0}}$.

Riesz interpolation theorem

- By the disjointness of the B_j 's we have (even when $q_0 = 1$)

$$\|g_{it}\|_{L^{q_0'}} = \|g\|_{L^{q'}}^{\frac{q'}{q_0'}}$$

since $\left| b_j^{Q(it)} \right| = b_j^{\frac{q'}{q_0'}}$.

- Thus Hölder's inequality and the hypothesis give

$$\begin{aligned} |F(it)| &\leq \|T(f_{it})\|_{L^{q_0}} \|g_{it}\|_{L^{q_0'}} \\ &\leq M_0 \|f_{it}\|_{L^{p_0}} \|g_{it}\|_{L^{q_0'}} \\ &= M_0 \|f\|_{L^p}^{\frac{p}{p_0}} \|g\|_{L^{q'}}^{\frac{q'}{q_0'}}. \end{aligned}$$

Riesz interpolation theorem

- By similar calculations, which are valid even when $p_1 = \infty$ and $q_1 = 1$, we have

$$\|f_{1+it}\|_{L^{p_1}} = \|f\|_{L^p}^{\frac{p}{p_1}},$$

and

$$\|g_{1+it}\|_{L^{q_1'}} = \|g\|_{L^{q'}}^{\frac{q'}{q_1'}}.$$

- Proceeding as on the previous slide we deduce that

$$|F(1+it)| \leq M_1 \|f\|_{L^p}^{\frac{p}{p_1}} \|g\|_{L^{q'}}^{\frac{q'}{q'}}.$$

- We observe that F is holomorphic in the open strip S and continuous on its closure. Also, F is bounded on the closed unit strip (by some constant that depends on f and g).

Riesz interpolation theorem

- Therefore, using the lemma from the previous remark, we deduce that

$$|F(z)| \le \left(M_0 \|f\|_{L^p}^{\frac{p}{p_0}} \|g\|_{L^{q'}}^{\frac{q'}{q_0'}} \right)^{1-\theta} \left(M_1 \|f\|_{L^p}^{\frac{p}{p_1}} \|g\|_{L^{q'}}^{\frac{q'}{q_1'}} \right)^{\theta}$$
$$= M_0^{1-\theta} M_1^{\theta} \|f\|_{L^p} \|g\|_{L^{q'}},$$

when $\operatorname{Re} z = \theta$. Observe that $P(\theta) = Q(\theta) = 1$ and hence

$$F(\theta) = \int_Y T(f) g d\nu$$

- Taking the supremum over all finitely simple functions g on Y with $L^{q'}$ norm less than or equal to one, we conclude the proof of the Riesz interpolation theorem. □

Young's convolution inequality

Theorem

If $1 \leq p, q, r \leq \infty$ *satisfy*

$$\frac{1}{q} + 1 = \frac{1}{p} + \frac{1}{r}.$$

Then for all functions $f \in L^p(\mathbb{R})$ *and* $g \in L^r(\mathbb{R})$ *we have*

$$\|f * g\|_{L^q(\mathbb{R})} \leq \|f\|_{L^p(\mathbb{R})} \|g\|_{L^r(\mathbb{R})},$$

where $f * g$ *is the convolution of* f *and* g*, which is defined by*

$$f * g(x) = \int_{\mathbb{R}} f(x-y)g(y)dx = \int_{\mathbb{R}} f(y)g(x-y)dx.$$

Young's convolution inequality

Proof: Fix a function g in $L^r(\mathbb{R})$ and let $T(f) = f * g$.

- The operator T maps $L^1(\mathbb{R})$ to $L^r(\mathbb{R})$ with norm at most $\|g\|_{L^r}$, (Why?).
- The operator T also maps $L^{r'}(\mathbb{R})$ to $L^\infty(\mathbb{R})$ with norm at most $\|g\|_{L^r}$, (Why?).
- The Riesz interpolation theorem gives that T maps $L^p(\mathbb{R})$ to $L^q(\mathbb{R})$ with norm at most the quantity

$$\|g\|_{L^r}^{\theta}\|g\|_{L^r}^{1-\theta} = \|g\|_{L^r},$$

where

$$\frac{1}{p} = \frac{1-\theta}{1} + \frac{\theta}{r'} \quad \text{and} \quad \frac{1}{q} = \frac{1-\theta}{r} + \frac{\theta}{\infty}.$$

- This completes the proof of Young's inequality. □