Lecture 10

Meromorphic functions and residues

MATH 503, FALL 2025

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Meromorphic functions

Definition

A function f is said to be **meromorphic** in an open set $\Omega \subseteq \mathbb{C}$ if there is a set $A \subset \Omega$ such that

(a) A has no limit point in Ω,

(b) $f \in H(\Omega \setminus A)$,

(c) f has a pole at each point of A.

Remarks

- Note that the possibility $A = \emptyset$ is not excluded. Thus every $f \in H(\Omega)$ is meromorphic in Ω.
- Note also that (a) implies that no compact subset of Ω contains infinitely many points of A, and that A is therefore at most countable. Since every open set in $\mathbb{C}$ is σ-compact.

Residues

Definition

Let $\Omega \subseteq \mathbb{C}$ be open. If f is a meromorphic function in Ω with the set of poles $A \subset \Omega$. If $a \in A$, and if

$$Q(z) = \sum_{k=1}^{m} c_k (z-a)^{-k}, \qquad (*)$$

is the principal part of f at a, (i.e., $f - Q$ has a removable singularity at $a \in A$), then the number c_1 is called the **residue** of f at a, and we write

$$c_1 = \mathrm{Res}(f; a). \qquad (**)$$

- If Γ is a cycle and $a \notin \Gamma^*$, then by (*) and the global Cauchy theorem, we have
$$\frac{1}{2\pi i}\int_\Gamma Q(z)dz = c_1 \,\mathrm{Ind}_\Gamma(a) = \mathrm{Res}(Q; a)\,\mathrm{Ind}_\Gamma(a). \qquad (***)$$

Residue theorem

Theorem

Suppose f is a meromorphic function in an open set $\Omega \subseteq \mathbb{C}$. Let A be the set of points in Ω at which f has poles. If Γ is a cycle in $\Omega \setminus A$ such that

$$\mathrm{Ind}_\Gamma(\alpha) = 0 \quad \textit{for all} \quad \alpha \notin \Omega, \tag{A}$$

then

$$\frac{1}{2\pi i} \int_\Gamma f(z)dz = \sum_{a \in A} \mathrm{Res}(f; a)\, \mathrm{Ind}_\Gamma(a). \tag{B}$$

Proof: Let $B = \{a \in A : \mathrm{Ind}_\Gamma(a) \neq 0\}$. Let W be the complement of Γ^*.

- Then $\mathrm{Ind}_\Gamma(z)$ is constant in each component V of W.
- If V is unbounded, or if V intersects Ω^c, then (A) implies that $\mathrm{Ind}_\Gamma(z) = 0$ for every $z \in V$. Since A has no limit point in Ω, we conclude that B is a finite set.

Residue theorem

- The sum in (B), though formally infinite, is therefore actually finite.
- Let $a_1, \ldots, a_n$ be the points of B, let $Q_1, \ldots, Q_n$ be the principal parts of f at $a_1, \ldots, a_n$, and put $g = f - (Q_1 + \cdots + Q_n)$. (If $B = \varnothing$, a possibility which is not excluded, then $g = f$.)
- Set $\Omega_0 = \Omega \setminus (A \setminus B)$. Since g has removable singularities at $a_1, \ldots, a_n$, the global Cauchy theorem applied to the function g and the open set Ω_0, shows that

$$\int_\Gamma g(z)dz = 0.$$

- Hence $f = g + \sum_{j=1}^n Q_j$ and

$$\frac{1}{2\pi i}\int_\Gamma f(z)dz = \sum_{i=1}^n \frac{1}{2\pi i}\int_\Gamma Q_k(z)dz = \sum_{k=1}^n \mathrm{Res}\,(Q_k; a_k)\,\mathrm{Ind}_\Gamma\,(a_k)$$

and since f and Q_k have the same residue at a_k, we obtain (B). □

Applications of residues

Theorem

Suppose γ *is a closed path in a region* $\Omega \subseteq \mathbb{C}$*, such that* $\mathrm{Ind}_\gamma(\alpha) = 0$ *for every* $\alpha \notin \Omega$*. Suppose also that* $\mathrm{Ind}_\gamma(\alpha) = 0$ *or* 1 *for ever* $\alpha \in \Omega \setminus \gamma^*$*, and let* $\Omega_1 = \{\alpha \in \mathbb{C} : \mathrm{Ind}_\gamma(\alpha) = 1\}$*. For any* $f \in H(\Omega)$ *let* N_f *be the number of zeros of* f *in* Ω_1*, counted according to their multiplicities.*

(a) *(Argument principle) If* $f \in H(\Omega)$ *and* f *has no zeros on* γ^* *then*

$$N_f = \frac{1}{2\pi i} \int_\gamma \frac{f'(z)}{f(z)} dz = \mathrm{Ind}_\Gamma(0), \tag{A}$$

where $\Gamma = f \circ \gamma$*.*

(b) *(Rouché's theorem) If also* $g \in H(\Omega)$ *and*

$$|f(z) - g(z)| < |f(z)| \quad \text{for all} \quad z \in \gamma^*, \tag{B}$$

then $N_g = N_f$*.*

Applications of residues

Remark

- Rouché's theorem says that two holomorphic functions have the same number of zeros in Ω_1 if they are close together on the boundary of Ω_1, in the sense of condition (B).

Proof: Consider $\varphi = f'/f$, a meromorphic function in Ω.

- If $a \in \Omega$ and f has a zero of order $m = m(a)$ at a, then we have $f(z) = (z-a)^m h(z)$, where h and $1/h$ are holomorphic in some neighborhood V of a. In $V \setminus \{a\}$, we have

$$\varphi(z) = \frac{f'(z)}{f(z)} = \frac{m}{z-a} + \frac{h'(z)}{h(z)},$$

since $f'(z) = m(z-a)^{m-1}h(z) + (z-a)^m h'(z)$.

- Thus

$$\operatorname{Res}(\varphi; a) = m(a).$$

Applications of residues

- Let $A = \{a \in \Omega_1 : f(a) = 0\}$. If our assumptions about the index of γ are combined with the residue theorem one obtains

$$\frac{1}{2\pi i}\int_\gamma \frac{f'(z)}{f(z)}dz = \sum_{a\in A}\mathrm{Res}(\varphi; a) = \sum_{a\in A} m(a) = N_f.$$

- This proves one half of (A). The other half is a matter of direct computation:

$$\begin{aligned}\mathrm{Ind}_\Gamma(0) &= \frac{1}{2\pi i}\int_\Gamma \frac{dz}{z} = \frac{1}{2\pi i}\int_0^{2\pi} \frac{\Gamma'(s)}{\Gamma(s)}ds \\ &= \frac{1}{2\pi i}\int_0^{2\pi} \frac{f'(\gamma(s))}{f(\gamma(s))}\gamma'(s)ds = \frac{1}{2\pi i}\int_\gamma \frac{f'(z)}{f(z)}dz = N_f\end{aligned}$$

The parameter interval of γ was here taken to be $[0, 2\pi]$.

Applications of residues

- In order to prove the Rouché theorem recall the following lemma:

Lemma

Let $\gamma_0, \gamma_1 : [0,1] \to \mathbb{C}$ *be closed paths. Let* $\alpha \in \mathbb{C}$*, if*

$$|\gamma_1(s) - \gamma_0(s)| < |\alpha - \gamma_0(s)| \quad \text{for all} \quad s \in I = [0,1],$$

then $\mathrm{Ind}_{\gamma_1}(\alpha) = \mathrm{Ind}_{\gamma_0}(\alpha)$.

- Condition (B) shows that g has no zero on γ^*. Hence (A) holds with g in place of f. Put $\Gamma_0 = g \circ \gamma$. Then it follows from (B) that

$$|\Gamma(t) - \Gamma_0(t)| < |\Gamma(t)| \quad \text{for all} \quad t \in [0,1].$$

- Then by the previous lemma with $\alpha = 0$ and (A), we obtain

$$N_g = \mathrm{Ind}_{\Gamma_0}(0) = \mathrm{Ind}_{\Gamma}(0) = N_f. \qquad \square$$

Example

Example

Determine the numbers of zeros of

$$z^{87} + 36z^{57} + 71z^4 + z^3 - z + 1$$

inside $|z| = 1$.

Solution: Take $g(z) = z^{87} + 36z^{57} + 71z^4 + z^3 - z + 1$, and $f(z) = 71z^4$.

- Then for $|z| = 1$, we have

$$\begin{aligned} |f(z) - g(z)| &= \left|z^{87} + 36z^{57} + z^3 - z + 1\right| \\ &\le 1 + 36 + 1 + 1 + 1 < 71 = |f(z)|. \end{aligned}$$

- Hence, by Rouché's theorem with $\Omega = \mathbb{C}$ and $\Omega_1 = D(0,1)$ and $\gamma = e^{it}$ for $t \in [0, 2\pi]$ we conclude that $g(z)$ has four zeros inside $\partial D(0,1)$ counted according to multiplicity. □

Example

Example

We will prove that

$$\lim_{A\to\infty}\int_{-A}^{A}\frac{\sin x}{x}e^{itx}dx=\begin{cases}\pi & \text{if } |t|<1,\\ 0 & \text{if } |t|>1.\end{cases} \qquad (*)$$

The limit in (*) is $\pi/2$ when $t=\pm1$.

Proof: Since the function

$$\frac{\sin z}{z}e^{itz}$$

is entire, its integral over $[-A,A]$ equals that over the path Γ_A obtained by going from $-A$ to -1 along the real axis, from -1 to 1 along the lower half of the unit circle, and from 1 to A along the real axis. This follows from Cauchy's theorem.

Example

- Γ_A avoids the origin, and we may therefore use the identity

$$2i \sin z = e^{iz} - e^{-iz}$$

  to see that

$$\int_{\Gamma_A} \frac{\sin z}{z} e^{itz} dz = \varphi_A(t+1) - \varphi_A(t-1), \qquad (**)$$

  where

$$\frac{1}{\pi}\varphi_A(s) = \frac{1}{2\pi i}\int_{\Gamma_A} \frac{e^{isz}}{z} dz.$$

- We now complete Γ_A to a closed path in two ways:
  - First, by the semicircle from A to $-Ai$ to $-A$;
  - Secondly, by the semicircle from A to Ai to $-A$.
- The function e^{isz}/z has a single pole, at $z = 0$, where its residue is 1.

Example

- Integrating over the closed path contained in the lower half plane, it follows that
$$\frac{1}{\pi}\varphi_A(s) = \frac{1}{2\pi}\int_{-\pi}^{0} \exp\left(isAe^{i\theta}\right) d\theta. \tag{L}$$
- Integrating over the closed path contained in the upper half plane, it follows that
$$\frac{1}{\pi}\varphi_A(s) = 1 - \frac{1}{2\pi}\int_{0}^{\pi} \exp\left(isAe^{i\theta}\right) d\theta. \tag{U}$$
- Note that
$$\left|\exp\left(isAe^{i\theta}\right)\right| = \exp(-As\sin\theta) \tag{5}$$
and that this is < 1 and tends to 0 as $A \to \infty$ if s and $\sin\theta$ have the same sign.

Example

- The dominated convergence theorem shows therefore that the integral in (L) tends to 0 if $s < 0$, and the one in (U) tends to 0 if $s > 0$.
- Thus

$$\lim_{A\to\infty} \varphi_A(s) = \begin{cases} \pi & \text{if } s > 0 \\ 0 & \text{if } s < 0. \end{cases}$$

- If we use (**) and apply this bound to $s = t + 1$ and to $s = t - 1$, we obtain (*), i.e.

$$\lim_{A\to\infty} \int_{-A}^{A} \frac{\sin x}{x} e^{itx} dx = \begin{cases} \pi & \text{if } |t| < 1, \\ 0 & \text{if } |t| > 1. \end{cases}$$

- Since $\varphi_A(0) = \pi/2$, the limit above is $\pi/2$ when $t = \pm 1$.
- Note that (*) gives the Fourier transform of $(\sin x)/x$. □

Example

Example

We prove that

$$\int_{-\infty}^{\infty} \frac{dx}{1+x^2} = \pi$$

by using contour integration.

Proof: Clearly $1/\left(1+x^2\right)$ is the derivative of $\arctan x$ and the conclusion readily follows. However, this is a good example to illustrate the residue calculus. So we provide a residue calculation that leads to another proof.

- Consider the function

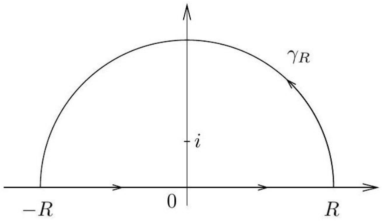
$$f(z) = \frac{1}{1+z^2},$$

  which is holomorphic in the complex plane except for simple poles at the points i and $-i$.

Example

- We choose the contour γ_R as below. The contour consists of the segment $[-R, R]$ on the real axis and of a large half-circle centered at the origin in the upper half-plane.
- Since we may write

$$f(z) = \frac{1}{(z-i)(z+i)} = \frac{1}{2i(z-i)} - \frac{1}{2i(z+i)},$$



we see that the residue of f at i is simply $1/2i$.

Example

- Therefore, if R is large enough, we have
$$\int_{\gamma_R} f(z)dz = \frac{2\pi i}{2i} = \pi.$$
- If we denote by C_R^+ the large half-circle of radius R, we see that
$$\left|\int_{C_R^+} f(z)dz\right| \leq \pi R \frac{B}{R^2} \leq \frac{M}{R},$$
where we have used the fact that $|f(z)| \leq B/|z|^2$ when $z \in C_R^+$ and R is large. So this integral goes to 0 as $R \to \infty$.
- Therefore, in the limit we find that
$$\int_{-\infty}^{\infty} f(x)dx = \pi$$
as desired. □

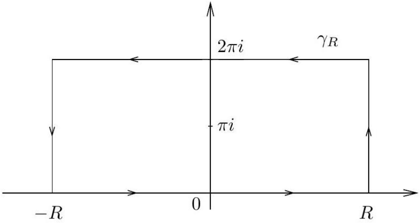
Example

Example

We prove that

$$\int_{-\infty}^{\infty} \frac{e^{ax}}{1+e^{x}}\,dx = \frac{\pi}{\sin \pi a}, \quad \text{with} \quad 0 < a < 1.$$

Proof: To prove this formula, let $f(z) = e^{az}/(1+e^{z})$, and consider the contour consisting of a rectangle in the upper half-plane with a side lying



on the real axis, and a parallel side on the line $\text{Im}(z) = 2\pi$.

Example

- The only point in the rectangle γ_R where the denominator of f vanishes is $z = \pi i$.
- To compute the residue of f at that point, we argue as follows:

$$(z - \pi i) f(z) = e^{az} \frac{z - \pi i}{1 + e^z} = e^{az} \frac{z - \pi i}{e^z - e^{\pi i}}$$

- On the right the inverse of a difference quotient, and in fact

$$\lim_{z \to \pi i} \frac{e^z - e^{\pi i}}{z - \pi i} = e^{\pi i} = -1$$

  since e^z is its own derivative.
- Therefore, the function f has a simple pole at πi with residue

$$\mathrm{Res}(f; \pi i) = -e^{a\pi i}.$$

Example

- As a consequence, the residue formula says that

$$\int_{\gamma_R} f = -2\pi i e^{a\pi i}.$$

  We now investigate the integrals of f over each side of the rectangle.

- Let I_R denote

$$\int_{-R}^{R} f(x)dx,$$

  and I the integral we wish to compute, so that $I_R \to I$ as $R \to \infty$.

- Then, it is clear that the integral of f over the top side of the rectangle (with the orientation from right to left) is

$$-e^{2\pi i a} I_R.$$

Example

- Finally, if $A_R = \{R + it : 0 \le t \le 2\pi\}$ denotes the vertical side on the right, then

$$\left|\int_{A_R} f\right| \le \int_0^{2\pi} \left|\frac{e^{a(R+it)}}{1+e^{R+it}}\right| dt \le Ce^{(a-1)R}$$

and since $a < 1$, this integral tends to 0 as $R \to \infty$.

- Similarly, the integral over the vertical segment on the left goes to 0, since it can be bounded by Ce^{-aR} and $a > 0$. Therefore, we obtain

$$I - e^{2\pi i a} I = \lim_{R\to\infty} \int_{\gamma_R} f = -2\pi i e^{a\pi i}$$

from which we deduce

$$I = -2\pi i \frac{e^{a\pi i}}{1 - e^{2\pi i a}} = \frac{2\pi i}{e^{\pi i a} - e^{-\pi i a}} = \frac{\pi}{\sin \pi a}$$

and the computation is complete. □

Example

Example

Now we calculate another Fourier transform, namely

$$\int_{-\infty}^{\infty} \frac{e^{-2\pi i x\xi}}{\cosh \pi x} dx = \frac{1}{\cosh \pi\xi}$$

where

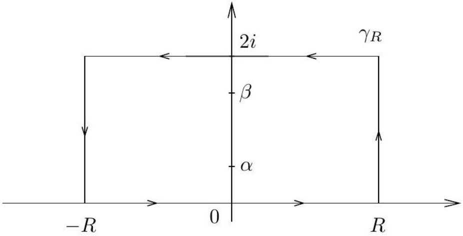
$$\cosh z = \frac{e^z + e^{-z}}{2}.$$

Proof: In other words, the function $1/\cosh \pi x$ is its own Fourier transform, a property also shared by $e^{-\pi x^2}$. To see this, we use a rectangle γ_R as below whose width goes to infinity, but whose height is fixed.

Example

- For a fixed $\xi \in \mathbb{R}$, let

$$f(z) = \frac{e^{-2\pi i z\xi}}{\cosh \pi z}.$$



- Note that the denominator of f vanishes precisely when $e^{\pi z} = -e^{-\pi z}$, that is, when $e^{2\pi z} = -1$.
- In other words, the only poles of f inside the rectangle are at the points $\alpha = i/2$ and $\beta = 3i/2$.

Example

- To find the residue of f at $\alpha = i/2$, we note that

$$\begin{aligned}(z-\alpha)f(z) &= e^{-2\pi i z\xi}\frac{2(z-\alpha)}{e^{\pi z}+e^{-\pi z}} \\ &= 2e^{-2\pi i z\xi}e^{\pi z}\frac{(z-\alpha)}{e^{2\pi z}-e^{2\pi\alpha}}\end{aligned}$$

- We recognize on the right the reciprocal of the difference quotient for the function $e^{2\pi z}$ at $z = \alpha$. Therefore

$$\lim_{z\to\alpha}(z-\alpha)f(z) = 2e^{-2\pi i\alpha\xi}e^{\pi\alpha}\frac{1}{2\pi e^{2\pi\alpha}} = \frac{e^{\pi\xi}}{\pi i},$$

  which shows that f has a simple pole at α with residue $e^{\pi\xi}/(\pi i)$.
- Similarly, we find that f has a simple pole at β with residue $-e^{3\pi\xi}/(\pi i)$.

Example

- We dispense with the integrals of f on the vertical sides by showing that they go to zero as R tends to infinity.
- Indeed, if $z = R + iy$ with $0 \le y \le 2$, then

$$\left| e^{-2\pi i z \xi} \right| \le e^{4\pi|\xi|},$$

and

$$\begin{aligned} |\cosh \pi z| &= \left| \frac{e^{\pi z} + e^{-\pi z}}{2} \right| \\ &\ge \frac{1}{2} \left| \left| e^{\pi z} \right| - \left| e^{-\pi z} \right| \right| \\ &\ge \frac{1}{2} \left(e^{\pi R} - e^{-\pi R} \right) \to \infty \quad \text{as} \quad R \to \infty. \end{aligned}$$

- This shows that the integral over the vertical segment on the right goes to 0 as $R \to \infty$. A similar argument shows that the integral of f over the vertical segment on the left also goes to 0 as $R \to \infty$.

Example

- Finally, we see that if I denotes the integral we wish to calculate, then the integral of f over the top side of the rectangle (with the orientation from right to left) is simply $-e^{4\pi\xi}I$, where we have used the fact that $\cosh \pi\zeta$ is periodic with period $2i$.
- In the limit as R tends to infinity, the residue formula gives

$$I - e^{4\pi\xi} I = 2\pi i \left(\frac{e^{\pi\xi}}{\pi i} - \frac{e^{3\pi\xi}}{\pi i} \right) = -2e^{2\pi\xi} \left(e^{\pi\xi} - e^{-\pi\xi} \right)$$

and since $1 - e^{4\pi\xi} = -e^{2\pi\xi}\left(e^{2\pi\xi} - e^{-2\pi\xi}\right)$, we find that

$$\begin{aligned} I &= 2\frac{e^{\pi\xi} - e^{-\pi\xi}}{e^{2\pi\xi} - e^{-2\pi\xi}} = 2\frac{e^{\pi\xi} - e^{-\pi\xi}}{\left(e^{\pi\xi} - e^{-\pi\xi}\right)\left(e^{\pi\xi} + e^{-\pi\xi}\right)} \\ &= \frac{2}{e^{\pi\xi} + e^{-\pi\xi}} = \frac{1}{\cosh \pi\xi} \end{aligned}$$

as claimed.

Example

Remark

- A similar argument actually establishes the following formula:

$$\int_{-\infty}^{\infty} e^{-2\pi i x\xi} \frac{\sin \pi a}{\cosh \pi x + \cos \pi a} dx = \frac{2 \sinh 2\pi a\xi}{\sinh 2\pi\xi}$$

  whenever $0 < a < 1$, and where $\sinh z = \left(e^{z} - e^{-z}\right)/2$.
- We have proved above the particular case $a = 1/2$.