Lecture 1

Complex Numbers

MATH 503, FALL 2025

September 4, 2025

Complex numbers

Definition (Complex numbers)

A **complex number** is an ordered pair $(a, b) \in \mathbb{R} \times \mathbb{R}$.

Definition (Addition and multiplication of complex numbers)

For two complex numbers $x = (a, b), y = (c, d) \in \mathbb{R} \times \mathbb{R}$ we define

- **addition** $+$ by setting

$$x + y = (a + c, b + d),$$

- **multiplication** $\cdot$ by setting

$$x \cdot y = (ac - bd, ad + bc).$$

Complex field

Theorem

These operations addition $+$ and multilpication $\cdot$ turn the set of all complex numbers into a field with $(0,0)$ and $(1,0)$ playing, respectively, the role of 0 and 1. This field will be denoted by $\mathbb{C}$.

Proof. We have to verify the field axioms.

Addition axioms (A)

- (A1) if $x, y \in \mathbb{C}$, then $x + y \in \mathbb{C}$,
- (A2) $x + y = y + x$ for all $x, y \in \mathbb{C}$,
- (A3) $(x + y) + z = x + (y + z)$ for all $x, y, z \in \mathbb{C}$,
- (A4) $\mathbb{C}$ contains the element 0 such that $x + 0 = x$ for all $x \in \mathbb{C}$,
- (A5) to every $x \in \mathbb{C}$ corresponds an element $(-x) \in \mathbb{C}$ such that

$$x + (-x) = 0.$$

Proof

Multiplication axioms (M)

- (M1) if $x, y \in \mathbb{C}$, then their product $xy \in \mathbb{C}$,
- (M2) $xy = yx$ for all $x, y \in \mathbb{C}$,
- (M3) $(xy)z = x(yz)$ for all $x, y, z \in \mathbb{C}$,
- (M4) $\mathbb{C}$ contains the element $1 \neq 0$ such that $1 \cdot x = x$ for all $x \in \mathbb{C}$,
- (M5) if $0 \neq x \in \mathbb{C}$ then there is an element $x^{-1} = \frac{1}{x} \in \mathbb{C}$ such that

$$x \cdot x^{-1} = 1.$$

Distributive law (D)

- (D1) $x(y + z) = xy + xz$ holds for all $x, y, z \in \mathbb{C}$.

Let $x = (a, b), y = (c, d), z = (e, f)$. We will use the field structure of $\mathbb{R}$.

- **Proof of (A1).** By the definition of addition

$$x + y = (a, b) + (c, d) = (a + c, b + d) \in \mathbb{C}.$$

Proof

- **Proof of (A2).**

$$x + y = (a + c, b + d) = (c + a) + (d + b) = y + x.$$

- **Proof of (A3).**

$$\begin{aligned}(x + y) + z &= (a + c, b + d) + (e, f)\\ &= (a + c + e, b + d + f)\\ &= (a, b) + (c + e, d + f) = x + (y + z).\end{aligned}$$

- **Proof of (A4).**

$$x + 0 = (a, b) + (0, 0) = (a, b) = x.$$

- **Proof of (A5).** Set $-x = (-a, -b)$ and note that

$$x + (-x) = (a - a, b - b) = (0, 0) = 0.$$

Proof

- **Proof of (M1).** By the definition of multiplication

$$x \cdot y = (a, b) \cdot (c, d) = (ac - bd, ad + bc) \in \mathbb{C}.$$

- **Proof of (M2).**

$$x \cdot y = (ac - bd, ad + bc) = (ca - db, da + cb) = y \cdot x.$$

- **Proof of (M3).**

$$\begin{aligned}(x \cdot y) \cdot z &= (ac - bd, ad + bc) \cdot (e, f) \\ &= (ace - bde - adf - bcf, acf - bdf + ade + bce) \\ &= (a, b) \cdot (ce - df, cf + de) = x \cdot (y \cdot z).\end{aligned}$$

- **Proof of (M4).**

$$1 \cdot x = (1, 0) \cdot (a, b) = (a, b) = x.$$

Proof

- **Proof of (M5).** If $x \neq 0$ then $(a, b) \neq (0, 0)$, which means that at least one of the real numbers a, b is different from 0. Hence $a^2 + b^2 > 0$ and we define
$$\frac{1}{x} = \left(\frac{a}{a^2 + b^2}, \frac{-b}{a^2 + b^2} \right).$$
Then
$$x \cdot \frac{1}{x} = (a, b) \cdot \left(\frac{a}{a^2 + b^2}, \frac{-b}{a^2 + b^2} \right) = (1, 0).$$
- **Proof of (D1).**
$$\begin{aligned} x \cdot (y + z) &= (a, b) \cdot (c + e, d + f) \\ &= (ac + ae - bd - bf, ad + af + bc + be) \\ &= (ac - bd, ad + bc) + (ae - bf, af + be) \\ &= x \cdot y + x \cdot z. \end{aligned}$$

This completes the proof that $\mathbb{C}$ is a field. □

Imaginary number i

Remark

For any $a, b \in \mathbb{R}$ we have

$$(a,0)+(b,0)=(a+b,0) \quad \text{and} \quad (a,0)\cdot(b,0)=(ab,0).$$

- The complex numbers from the set $\{(a,0) : a \in \mathbb{R}\}$ have the same arithmetic properties as the corresponding real numbers $\mathbb{R}$.
- We can therefore identify $(a,0)$ with a. This identification gives us the real field $\mathbb{R}$ as a subfield of the complex field $\mathbb{C}$.
- We have defined the complex numbers $\mathbb{C}$ without any reference to the mysterious square root of -1. We now show that the notation (a,b) is equivalent to the more customary $a+bi$.

Definition

We define the **imaginary number** by setting $i=(0,1)$.

Equivalent definition of $\mathbb{C}$

Theorem

One has that $i^2 = -1$.

Proof.

Note that $i^2 = (0,1) \cdot (0,1) = (-1,0)$. □

Theorem

We also have

$$\mathbb{C} = \{a + ib : a, b \in \mathbb{R}\}.$$

Proof.

It suffices to note that

$$\begin{aligned} a + ib &= (a,0) + (0,1) \cdot (b,0) \\ &= (a,0) + (0,b) = (a,b). \end{aligned}$$

□

Conjugate, real and imaginary parts

Definition

If $z \in \mathbb{C}$ and $z = a + ib$ for some $a, b \in \mathbb{R}$ then the complex number

$$\overline{z} = a - ib$$

is called the **conjugate** of z. The numbers a and b are the **real part** and **imaginary part** of z respectively. We shall write

$$a = \Re(z) = \mathrm{Re}(z) \quad \text{and} \quad b = \Im(z) = \mathrm{Im}(z).$$

Theorem

If $z, w \in \mathbb{C}$ then

(i) $\overline{z + w} = \overline{z} + \overline{w}$.

(ii) $\overline{zw} = \overline{z} \cdot \overline{w}$.

(iii) $z + \overline{z} = 2\mathrm{Re}(z)$ *and* $z - \overline{z} = 2i\mathrm{Im}(z)$.

(iv) $z\overline{z}$ *is a positive real number except when* $z = 0$.

Proof

Proof. Let $z = a + ib$ and $w = c + id$.

- **Proof of (i).** Note that

$$\overline{z + w} = \overline{(a + c) + i(b + d)} = (a + c) - i(b + d) = \overline{z} + \overline{w}.$$

- **Proof of (ii).** Note that

$$\overline{z \cdot w} = (ac - bd) - i(ad + bc) \quad \text{and}$$
$$\overline{z} \cdot \overline{w} = (a - ib)(c - id) = (ac - bd) - i(ad + bc).$$

- **Proof of (iii).** We have

$$z + \overline{z} = (a + ib) + (a - ib) = 2a = 2\mathrm{Re}(z),$$
$$z - \overline{z} = (a + ib) - (a - ib) = 2ib = 2i\mathrm{Im}(z).$$

- **Proof of (iv).** We have $z \cdot \overline{z} = (a + ib)(a - ib) = a^2 + b^2 > 0$ if and only if $z \neq 0$. □

Absolute value on $\mathbb{C}$

Definition

If $z \in \mathbb{C}$ its **absolute value** $|z|$ is defined by setting

$$|z| = \sqrt{z \cdot \overline{z}}.$$

Remark

This absolute value exists and is unique. Moreover, it coincides with the absolute value from $\mathbb{R}$. If $x \in \mathbb{R}$ then $\overline{x} = x$ hence $|x| = \sqrt{x \cdot \overline{x}} = \sqrt{x^2}$. Thus

$$|x| = \begin{cases} x & \text{if } x \geq 0, \\ -x & \text{if } x < 0. \end{cases}$$

Properties of the absolute value on $\mathbb{C}$

Theorem

If $z, w \in \mathbb{C}$ then

(i) $|z| > 0$ *if and only if* $z \neq 0$, *and* $|0| = 0$.

(ii) $|\overline{z}| = |z|$.

(iii) $|zw| = |z||w|$.

(iv) $|\mathrm{Re}(z)| \leq |z|$ *and* $|\mathrm{Im}(z)| \leq |z|$

(v) $|z + w| \leq |z| + |w|$.

Proof. Let $z = a + ib$ and $w = c + id$.

- **Proof of (i).** From the previous theorem we have

$$|z|^2 = z \cdot \overline{z} = (a + ib)(a - ib) = a^2 + b^2 > 0,$$

which gives the desired claim.

Proof

- **Proof of (ii).** Note that $|z|^2 = a^2 + b^2 = |\overline{z}|^2$.
- **Proof of (iii).** Note that

$$|z \cdot w| = (ac - bd)^2 + (ad + bc)^2 = (a^2 + b^2)(c^2 + d^2) = |z|^2|w|^2.$$

- **Proof of (iv).** We have

$$|\mathrm{Re}(z)| = |a| \leq \sqrt{a^2 + b^2} = |z|, \quad \text{and} \quad |\mathrm{Im}(z)| = |b| \leq \sqrt{a^2 + b^2} = |z|.$$

- **Proof of (v).** Note that $\overline{z}w$ is the conjugate of $z\overline{w}$ so that $z\overline{w} + \overline{z}w = 2\mathrm{Re}(z\overline{w})$. Hence

$$\begin{aligned} |z + w|^2 &= (z + w)(\overline{z} + \overline{w}) = z\overline{z} + z\overline{w} + \overline{z}w + w\overline{w} \\ &= |z|^2 + 2\mathrm{Re}(z\overline{w}) + |w|^2 \\ &\leq |z|^2 + 2|\mathrm{Re}(z\overline{w})| + |w|^2 \\ &\leq |z|^2 + 2|z||w| + |w|^2 = (|z| + |w|)^2. \end{aligned}$$

The proof of the theorem is completed. □

An argument of a complex number

Definition

An **argument** $\arg(z)$ of $z = a + ib \in \mathbb{C}$ is defined as the angle which the line segment from $(0,0)$ to (a,b) makes with the positive real axis. The argument is not unique, but is determined up to a multiple of 2π.

If $r = |z|$ and $\theta = \arg(z)$ is an argument of $z \in \mathbb{C}$, we may write

$$z = r(\cos\theta + i\sin\theta).$$

Then for $z_1, z_2, z \in \mathbb{C}$ it follows from trigonometric identities that

$$\begin{aligned}\arg(z_1 z_2) &= \arg(z_1) + \arg(z_2),\\ \arg(z_1/z_2) &= \arg(z_1) - \arg(z_2) \quad \text{if} \quad z_2 \neq 0,\\ \arg(\overline{z}) &= -\arg(z).\end{aligned}$$

The argument of z is called **principal** if $\arg(z) \in (-\pi, \pi]$.

Convergence in $\mathbb{C}$

Definition

- We say that a sequence of complex numbers $(z_n)_{n\in\mathbb{N}} \subseteq \mathbb{C}$ **converges** to $z \in \mathbb{C}$ and write $\lim_{n\to\infty} z_n = z$ if and only if

$$\lim_{n\to\infty} |z_n - z| = 0.$$

- This is also equivalent to say that for every $\varepsilon > 0$ there exists an integer $N_\varepsilon \in \mathbb{N}$ such that if $n \geq N_\varepsilon$ then

$$|z_n - z| < \varepsilon.$$

- Obviously $\lim_{n\to\infty} z_n = z$ iff $\lim_{n\to\infty} \mathrm{Re}(z_n) = \mathrm{Re}(z)$ and $\lim_{n\to\infty} \mathrm{Im}(z_n) = \mathrm{Im}(z)$.
- We say that $z_n \xrightarrow[n\to\infty]{} \infty$ **diverges** iff $\lim_{n\to\infty} |z_n| = \infty$.

Complex plane $\mathbb{C}$ is a complete metric space

Definition

- We say that a sequence of complex numbers $(z_n)_{n\in\mathbb{N}} \subseteq \mathbb{C}$ is said to be a **Cauchy sequence** in $\mathbb{C}$ (or simply **Cauchy**) iff

$$\lim_{m,n\to\infty} |z_n - z_m| = 0.$$

- This is also equivalent to say that for every $\varepsilon > 0$ there exists an integer $N_\varepsilon \in \mathbb{N}$ such that if $m, n \geq N_\varepsilon$ then

$$|z_n - z_m| < \varepsilon.$$

Theorem

The complex plane $\mathbb{C}$ with a metric given by

$$d(z_1, z_2) = |z_1 - z_2| \quad \textit{for} \quad z_1, z_2 \in \mathbb{C}$$

is a complete metric space.

Discs, punctured discs, circles in $\mathbb{C}$

- If $a \in \mathbb{C}$ and $r > 0$, we define the **open disc** $D(a, r)$ of radius r centered at a to be the set of the form
$$D(a, r) = \{z \in \mathbb{C} : |z - a| < r\}.$$
- We write $D = D(0, 1)$ for the open unit disc centered at the origin.
- The **closed disc** $\overline{D}(a, r)$ of radius r centered at a is defined by
$$\overline{D}(a, r) = \{z \in \mathbb{C} : |z - a| \le r\}.$$
- The **punctured disc** $D'(a, r)$ of radius r centered at a is defined by
$$D'(a, r) = \{z \in \mathbb{C} : 0 < |z - a| < r\}.$$
We observe that it is an open set.
- The **circle** $C(a, r)$ of radius r centered at a is defined by
$$C(a, r) = \{z \in \mathbb{C} : |z - a| = r\} = \overline{D}(a, r) \setminus D(a, r).$$

Topology of $\mathbb{C}$

- A set $\Omega \subseteq \mathbb{C}$ is **open** if for every $a \in \Omega$ there exists $r > 0$ such that

$$D(a, r) \subseteq \Omega.$$

- A set Ω is **closed** if its complement $\Omega^c = \mathbb{C} \setminus \Omega$ is open.
- This property can be reformulated in terms of limit points. A point $z \in \mathbb{C}$ is said to be a **limit point** of the set Ω if there exists a sequence of points $(z_n)_{n \in \mathbb{N}} \subseteq \Omega$ such that $z_n \neq z$ and $\lim_{n \to \infty} z_n = z$.
- One can check that a set is closed if and only if it contains all its limit points. The **closure** of any set Ω is the union of Ω and its limit points, and is often denoted by $\overline{\Omega}$.
- Finally, the **boundary** of a set Ω is equal to its closure minus its interior, and is often denoted by $\partial\Omega$.
- For instance the circle $C(a, r)$ is the boundary of the disc $D(a, r)$.

Polygons in $\mathbb{C}$

- If a and b are complex numbers, $[a, b]$ denotes the closed **line segment** with endpoints a and b.
- If t_1 and t_2 are arbitrary real numbers with $t_1 < t_2$, then we may write

$$[a, b] = \left\{ a + \frac{t - t_1}{t_2 - t_1}(b - a) : t_1 \leq t \leq t_2 \right\}$$

- The notation is extended as follows. If $a_1, a_2, \ldots, a_{n+1}$ are points in $\mathbb{C}$, a **polygon** from a_1 to a_{n+1} (or a polygon joining a_1 to a_{n+1}) is defined as

$$\bigcup_{j=1}^{n} [a_j, a_{j+1}] ,$$

often abbreviated as $[a_1, \ldots, a_{n+1}]$.

Connectedness

- Let (X, d) be a metric space and $E \subseteq X$.

Definition

A set E is **connected** if E cannot be written as a disjoint union of two non-empty relative open subsets of E.

- Thus $E = A \cup B$ with $A \cap B = \emptyset$ and A, B open in E implies that either $A = \emptyset$ or $B = \emptyset$. Otherwise $E = A \cup B$ is called a separation E into open sets.
- Union E of two disjoint open discs A and B is not connected since

$$E = A \cup B = (A \cap E) \cup (B \cap E),$$

  and $A \cap E$ and $B \cap E$ are non-empty, disjoint and relatively open in E.
- An open connected set in a metric space is called a **region**.

Connected components

Definition

A maximal connected subset of E is called a component of E

- For $a \in E$, let $C(a)$ be the union of all connected subsets of E containing a. We observe that $a \in C(a)$ since $\{a\}$ is connected and

$$E = \bigcup_{a \in E} C(a).$$

Lemma

(i) *$C(a)$ is connected.*

(ii) *The components of E are either disjoint or identical.*

(iii) *The components of an open set are open.*

By combining (i), (ii) and (iii), we conclude:

Theorem

An open set in a metric space is a disjoint union of regions.

Connected components

Proof of (i)

- We first prove that $C(a)$ is connected. The proof is by contradiction.
- Let $C(a) = A \cup B$ be a separation of $C(a)$ into open sets.
- We may assume that $a \in A$ and $b \in B$. Then, since $b \in C(a)$ and $C(a)$ is the union of all connected subsets of E containing a, there exists $E_0 \subseteq E$ such that $E_0 \subseteq C(a)$ is connected and $a, b \in E_0$.
- Thus

$$E_0 = E_0 \cap C(a) = E_0 \cap (A \cup B) = (E_0 \cap A) \cup (E_0 \cap B)$$

implies that either $E_0 \cap A = \emptyset$ or $E_0 \cap B = \emptyset$.

- This is a contradiction since $a \in E_0 \cap A$ and $b \in E_0 \cap B$. □

- Thus every component of E is of the form $C(a)$ with $a \in E$.

Connected components

Proof of (ii)

- The components of E are either disjoint or identical. Let $a, b \in E$.
- Assume that $C(a) \cap C(b) \neq \emptyset$. We prove that $C(a) = C(b)$.
- Let $x \in C(a) \cap C(b)$. Then $x \in C(a)$. Since $C(a)$ is connected, we derive that $C(a) \subseteq C(x)$. Then $a \in C(x)$ which implies $C(x) \subseteq C(a)$ since $C(x)$ is connected. Thus $C(a) = C(x)$.
- Similarly $C(b) = C(x)$ and hence $C(a) = C(b)$. □

Proof of (iii)

- The components of an open set are open. Let E be an open set.
- It suffices to show that $C(a)$ with $a \in E$ is open. Let $x \in C(a)$.
- Then $C(x) = C(a)$ by (ii). Since $x \in E$ and E is open, then $D(x, r) \subseteq E$ for some $r > 0$. In fact $D(x, r) \subseteq C(x)$ since $D(x, r)$ is connected containing x. Thus $x \in D(x, r) \subseteq C(a)$ and hence $C(a)$ is open as desired. □

Path connectedness

Theorem

Let E be a non-empty open subset of $\mathbb{C}$. Then E is connected if and only if any two points in E can be joined by a polygonal path that lies in E.

Proof ($\Longrightarrow$).

- Assume that E is connected. Since $E \neq \emptyset$, let $a \in E$. Let E_1 be the subset of all elements of E that can be joined to a by a polygonal path. Let E_2 be the complement of E_1 in E. Then
$$E = E_1 \cup E_2 \quad \text{with} \quad E_1 \cap E_2 = \emptyset, \quad \text{and} \quad a \in E_1.$$
- It suffices to show that both E_1 and E_2 are open subsets of E.
- Then $E_2 = \emptyset$ since E is connected and $a \in E_1$. Thus every point of E can be joined to a by a polygonal path that lies in E. Hence any two points of E can be joined by a polygonal path that lies in E via a.

Path connectedness

- First, we show that E_1 is open. Let $a_1 \in E_1$, then $a_1 \in E$ and since E is open, we have $D(a_1, r_1) \subseteq E$ for some $r_1 > 0$.
- Any point of $D(a_1, r_1)$ can be joined to a_1 and hence to a by a polygonal path that lies in E since $a_1 \in E_1$. Thus

$$a_1 \in D(a_1, r_1) \subseteq E_1.$$

- Next, we show that E_2 is open. Let $a_2 \in E_2$. Again we find $r_2 > 0$ such that $D(a_2, r_2) \subseteq E$ since E is open.
- Now, as above, we see that no point of this disc can be joined to a as $a_2 \in E_2$ and hence $a_2 \in D(a_2, r_2) \subseteq E_2$. □

Proof ($\Longleftarrow$).

- Now we assume that any two points of E can be joined by a polygonal path in E and we show that E is connected.

Path connectedness

- Let $E = E_1 \cup E_2$ be a separation of E into open sets.
- Let $a_1 \in E_1$ and $a_2 \in E_2$ be such that
$$\chi(t) = ta_1 + (1-t)a_2 \quad \text{with} \quad 0 < t < 1$$
is an open segment from a_2 to a_1 lying in E.
- Let
$$V = \{t \in (0,1) \mid \chi(t) \in E_1\} \quad \text{and} \quad W = \{t \in (0,1) \mid \chi(t) \in E_2\}.$$
- We see that V and W are open in $(0,1)$. Further, we have separation of the open interval $(0,1)$ into open sets
$$(0,1) = V \cup W, \quad V \cap W = \emptyset.$$
- Since $a_1 \in E_1$ and E_1 is open, then $D(a_1, r_3) \subseteq E_1$ for some $r_3 > 0$. This implies $V \neq \emptyset$. Similarly $W \neq \emptyset$.
- Hence the interval $(0,1)$ is not connected. This is a contradiction. □